SYSTEMS OF LOGIC BASED ON ORDINALS

A. M. Turing

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CONTENTS

The well known theorem of Gödel shows that every system of logic is in a certain sense incomplete, but at the same time it indicates means whereby from a system L of logic a more complete system L' may be obtained. By repeating the process we get a sequence $L, L_1 = L', L_2 = L_1', L_3 = L_2'$,... of logics each more complete than the preceding. A logic L_ω may then be constructed in which the provable theorems are the totality of theorems provable with the help of the logics L, L_1, L_2 ,... We may then form $L_{2\omega}$ related to L_ω in the same way as L_ω was related to L. Proceeding in this way we can associate a system of logic with any given constructive ordinal.[1] It may be asked whether a sequence of logics of this kind is complete in the sense that to any problem A there corresponds an ordinal α such that A is solvable by means of the logic L_α. I propose to investigate this problem in a rather more general case, and to give some other examples of ways in which systems of logic may be associated with constructive ordinals.

[1] The situation is not quite so simple as is suggested by this crude argument. See pages 44-48.

1. The calculus of conversion. Gödel representations.

It will be convenient to be able to use the 'conversion calculus' ~~for~~ of Church for the description of functions and some other purposes. This will make greater clarity and simplicity of expression possible. I shall give a short account of this calculus. For more detailed descriptions see Church [3], [2], Kleene [1], Church and Rosser [1].

The formulae of the calculus are formed from the symbols $\{$, $\}$ $($, $)$, $[$, $]$, λ, δ, and an infinite list of others called variables; we shall take for our infinite list $a, b, \ldots, z, x', x'', \ldots$ Certain finite sequences of such symbols are called <u>well-formed formulae</u> (abbreviated to W.F.F.); we shall define this class inductively, and simultaneously define the free and the bound variables of a W.F.F. Any variable is a W.F.F.; it is its only free variable, and it has no bound variables. δ is a W.F.F. and has no free or bound variables. If $\underline{M}$ and $\underline{N}$ are W.F.F. then $\{\underline{M}\}(\underline{N})$ is a W.F.F. whose free variables are the free variables of $\underline{M}$ together with the free variables of $\underline{N}$, and whose bound variables are the bound variables of $\underline{M}$ together with those of $\underline{N}$. If $\underline{M}$ is a W.F.F. and $\underline{V}$ one of its free variables, then $\lambda\underline{V}[\underline{M}]$ is a W.F.F. whose free variables are those of $\underline{M}$ with the exception of $\underline{V}$, and whose bound variables are those of $\underline{M}$ together with $\underline{V}$. No sequence of symbols is a W.F.F. except in consequence of these three statements.

In metamathematical statements we shall use underlined letters to stand for variable or undetermined formulae, as was done in the last paragraph, and in future such letters will stand for well-formed formulae unless otherwise stated. Small letters underlined will stand for formulae representing undetermined positive integers (see below).

A W.F.F. is said to be in normal form if it has no parts of the form $\{\lambda\underline{V}[\underline{M}]\}(\underline{N})$ and none of the form $\{\{\delta\}(\underline{M})\}(\underline{N})$ where $\underline{M}$ and $\underline{N}$ have no free variables.

We say that one W.F.F. is <u>immediately convertible</u> into another if it is obtained from it either by

(i) Replacing one occurrence of a well-formed part $\lambda \underline{V}[\underline{M}]$ by $\lambda \underline{U}[\underline{N}]$, where the variable $\underline{U}$ does not occur in $\underline{M}$, and $\underline{N}$ is obtained from $\underline{M}$ by replacing the variable $\underline{V}$ by $\underline{U}$ throughout.

(ii) Replacing a well-formed part $\{\lambda \underline{V}[\underline{M}]\}(\underline{N})$ by the formulae which is obtained from $\underline{M}$ by replacing $\underline{V}$ by $\underline{N}$ throughout, provided that the bound variables of $\underline{M}$ are distinct both from $\underline{V}$ and from the free variables of $\underline{N}$.

(iii) The converse process of ii.

(iv) Replacing a well-formed part $\{\{\delta\}(\underline{M})\}(\underline{M})$ by $\lambda f[\lambda x[\{f\}(\{f\}(x))]]$ if $\underline{M}$ is in normal form and has no free variables.

(v) Replacing a well-formed part $\{\{\delta\}(\underline{M})\}(\underline{N})$ by $\lambda f[\lambda x[\{f\}(x)]]$ if $\underline{M}$ and $\underline{N}$ are in normal form and not transformable into one another by repeated application of i, and have no free variables.

(vi) The converse process of iv.

(vii) The converse process of v.

These rules could have been expressed in such a way that in no case could there be any doubt as to the admissibility or the result of the transformation (in particular this can be done in the case of process v.).

A formula $\underline{A}$ is said to be <u>convertible</u> into another $\underline{B}$ (abbreviated to '$\underline{A}$ conv $\underline{B}$') if there is an finite chain of immediate conversions leading from one formula to the other. It is easily seen that the relation of convertibility is an equivalence relation, i.e. it is symmetric, transitive and reflexive.

Since the formulae are liable to be very lengthy we need means for abbreviating them. If we wish to introduce a particular letter as an abbreviation for a particular lengthy formula we shall write the letter followed by '$\rightarrow$' and then by the formula, thus

$$\underline{I} \rightarrow \lambda x[x]$$

indicates that $\underline{I}$ is an abbreviation for $\lambda x[x]$. We shall also use the arrow in less sharply defined senses, but never so as to cause any real confusion. In these cases the meaning of the arrow may be rendered by the words 'stands for'.

If a formula $\underline{F}$ is, or is represented by, a single symbol we abbreviate $\{\underline{F}\}(\underline{X})$ to $\underline{F}(\underline{X})$. A formula $\{\{\underline{F}\}(\underline{X})\}(\underline{Y})$ may be abbreviated to $\{\underline{F}\}(\underline{X},\underline{Y})$, or to $\underline{F}(\underline{X},\underline{Y})$ if $\underline{F}$ is, or is represented by, a single symbol. Similarly for $\{\{\{\underline{F}\}(\underline{X})\}(\underline{Y})\}(\underline{Z})$ etc. A formula $\lambda \underline{V}_1[\lambda \underline{V}_2 \ldots [\lambda \underline{V}_r[\underline{M}]]\ldots]$ may be abbreviated to $\lambda \underline{V}_1 \underline{V}_2 \ldots \underline{V}_r . \underline{M}$.

We have not as yet assigned any meanings to our formulae, and we do not intend to do so in general. An exception may be made for the case of the positive integers which are very conveniently represented by the formulae $\lambda f x . f(x)$, $\lambda f x . f(f(x))$, ... In fact we introduce the abbreviations

$$1 \rightarrow \lambda f x . f(x)$$
$$2 \rightarrow \lambda f x . f(f(x))$$
$$3 \rightarrow \lambda f x . f(f(f(x)))$$

etc.

and also say for example that $\lambda f x . f(f(x))$ (in full $\lambda f[\lambda x[\{f\}(\{f\}(x))]]$) represents the positive integer 2. Later we shall allow certain formulae to represent ordinals, but otherwise we leave them without explicit meaning; an implicit meaning may be suggested by the abbreviations used. In any case where any meaning is assigned to formulae it is desirable that the meaning be invariant under conversion. Our definitions of the positive integers do not violate this requirement, as it may be proved that no two formulae representing different positive integers are convertible into one another.

In connection with the positive integers we introduce the abbreviation

$$S \rightarrow \lambda u f x . f(u(f, x))$$

This formula has the property that if $\underline{n}$ represents a positive integer $S(\underline{n})$ is convertible to a formula representing its successor.[2]

[2] This follows from (A) below.

Formulae representing undetermined positive integers will be represented by small letters underlined, and we shall adopt once for all the convention that if a letter, n say, stands for a positive integer, then the same letter underlined, $\underline{n}$, stands for the formula representing the positive integer. When no confusion arises from doing so we shall omit to distinguish between an integer and the formula which represents it.

Suppose $f(n)$ is a function of positive integers taking positive integers as values, and that there is a W.F.F. $\underline{F}$ not containing δ such that for each positive integer n, $\underline{F}(\underline{n})$ is convertible to the formula representing $f(n)$. We shall then say that $f(n)$ is λ -definable or formally definable, and that $\underline{F}$ formally defines $f(n)$. Similar conventions are used for functions of more than one variable. The sum function is for instance formally defined by $\lambda abfx . a(f, b(f, x))$; in fact for any positive integers m, n, p for which $m + n = p$ we have

$\{\lambda abfx . a(f, b(f, x))\}(\underline{m}, \underline{n})$ conv $\underline{p}$

In order to emphasize this relation we introduce the abbreviation

$$\underline{X} + \underline{Y} \rightarrow \{\lambda abfx . a(f, b(f, x))\}(\underline{X}, \underline{Y})$$

and will use similar notations for sums of three or more terms, products etc.

For any W.F.F. $\underline{G}$ we shall say that $\underline{G}$ enumerates the sequence $\underline{G}(\underline{1})$, $\underline{G}(\underline{2})$, . . . and any other sequence whose terms are convertible to those of this sequence.

When a formula is convertible to another which is in normal form the second is described as a normal form of the first, which is then said to have a normal form. I quote here some of the more important theorems concerning normal forms.

(A) If a formula has two normal forms they are convertible into one another by the use of (i) alone. (Church and Rosser [1], 479, 481).

(B) If a formula has a normal form then every well-formed part of it has a normal form. (Church and Rosser [1], 480-481).

(C) There is (demonstrably) no process whereby one can tell of a formula whether it has a normal form. (Church [3], 360, Theorem XVIII.)

We often need to be able to describe formulae by means of positive integers. The method used here is due to Gödel (Gödel [1]). To each single symbol S of the calculus we assign an integer $r[S]$ as in the table below.

S	{, (or [	},) or]	λ	δ	a	...	z	x'	x''	x'''	...
$r[S]$	1	2	3	4	5	...	30	31	32	33	...

If $S_1 S_2 \ldots S_k$ is a sequence of symbols then $2^{r[S_1]} 3^{r[S_2]} \ldots p_k^{r[S_k]}$ (where p_k is the k th prime number) is called the <u>Gödel representation</u> (G.R.) of that sequence of symbols. No two W.F.F. have the same G.R.

Two theorems on G.R. of W.F.F. are quoted here.

(D) There is a W.F.F. form such that if a is the G.R. of a W.F.F. $\underline{A}$ without free variables then $\mathrm{form}(\underline{a})$ conv $\underline{A}$. (This follows from a similar theorem to be found in Church [3], 53-66. Metads are used there in place of G.R.)

(E) There is a W.F.F. $\mathfrak{Gr}$ such that if $\underline{A}$ is a W.F.F. with a normal form without free variables, then $\mathfrak{Gr}(\underline{A})$ conv $\underline{a}$, where a is the G.R. of a normal form of $\underline{A}$. (Church [3], 53-66, as (D)).

2. Effective calculability. Abbreviation of treatment.

A function is said to be 'effectively calculable' if its values can be found by some purely mechanical process. Although it is fairly easy to get an intuitive grasp of this idea it is nevertheless desirable to have some more definite, mathematically expressible definition. Such a definition was first given by Gödel at Princeton in 1934 (Gödel [2], 26) following in part an unpublished suggestion of Herbrand, and has since been developed by Kleene (Kleene [2]). We shall not be concerned much here with this particular definition. Another definition of effective calculability has been given by Church (Church [3], 356-358) who identifies it with λ-definability. The author has recently suggested a definition corresponding more closely to the intuitive idea (Turing [1], see also Post [1]). It was said above "a function is effectively calculable if its values can be found by some purely mechanical process." We may take this statement literally, understanding by a purely mechanical process one which could be carried out by a machine. It is possible to give a mathematical description, in a certain normal form, of the structures of these machines. The development of these ideas leads to the author's definition of a computable function, and an identification of computability[3] with effective calculability.

[3] We shall use the expression 'computable function' to mean a function calculable by a machine, and let 'effectively calculable' refer to the intuitive idea without particular identification with any one of these definitions. We do not restrict the values taken by a computable function to be natural numbers; we may for instance have computable propositional functions.

It is not difficult though somewhat laborious, to prove these three definitions equivalent (Kleene [3], Turing [2]).

In the present paper we shall make considerable use of Church's identification of effective calculability with λ-definability, or, what comes to the same, of the identification with computability and one of the equivalence theorems. In most cases where we have to deal with an effectively calculable function we shall introduce the corresponding W.F.F. with some such phrase as "the function f is effectively calculable, let F be a formula λ-defining it" or "let F be a formula such that $F(\underline{n})$ is convertible to . . . whenever $\underline{n}$ represents a positive integer". In such cases there is no difficulty in seeing how a machine could in principle be designed to calculate the values of the function concerned, and assuming this done the equivalence theorem can be applied. A statement as to what the formula F actually is may be omitted. We may introduce immediately on this basis a W.F.F. $\mathfrak{D}$ with the property that $\mathfrak{D}(\underline{m}, \underline{n})$ conv $\underline{r}$ if r is the greatest positive integer for which m^r divides n, if any, and is 1 if there is none. We also introduce $\mathfrak{Dt}$ with the properties

$$\mathfrak{Dt}(\underline{n}, \underline{n}) \text{ conv } 3$$
$$\mathfrak{Dt}(\underline{n}+\underline{m}, \underline{n}) \text{ conv } 2$$
$$\mathfrak{Dt}(\underline{n}, \underline{n}+\underline{m}) \text{ conv } 1$$

There is another point to be made clear in connection with the point of view we are adopting. It is intended that all proofs that are given should be regarded no more critically than proofs in classical analysis. The subject matter, roughly speaking, is constructive systems of logic, but as the purpose is directed towards choosing a particular constructive system of logic for practical use; an attempt at this stage to put our theorems into constructive form would be putting the cart before the horse.

Those computable functions which take only the values 0 and 1 are of particular importance since they determine and are determined by computable properties, as may be seen by replacing '0' and '1' by 'true' and 'false'. But besides this type of property we may have to consider a different type, which is, roughly speaking, less constructive than the computable properties, but more so than the general predicates of classical mathematics. Suppose we have a computable function of the natural numbers taking natural numbers as values, then corresponding to this function there is the property of being a value of the function. Such a property we shall describe as 'axiomatic'; the reason for using this term is that it is possible to define such a property by giving a set of axioms, the property to hold for a given argument if and only if it is possible to deduce that it holds from the axioms.

Axiomatic properties may also be characterized in this way. A property ψ of positive integers is axiomatic if and only if there is a computable property φ of two positive integers, such that $\psi(x)$ is true if and only if there is a positive integer y such that $\varphi(x,y)$ is true. Or again ψ is axiomatic if and only if there is a W.F.F. $\underline{F}$ such that $\psi(n)$ is true if and only if $\underline{F}(\underline{n})$ conv 2.

3. Number theoretic theorems

By a <u>number theoretic theorem</u>[4] we shall mean a theorem of the

[4] I believe there is no generally accepted meaning for this term, but it should be noticed that we are using it in a rather restricted sense. The most generally accepted meaning is probably this: suppose we take an arbitrary formula of the function calculus of first order and replace the function variables by primitive recursive relations. The resulting formula represents a typical number theoretic theorem in this (more general) sense.

form ' $\theta(x)$ vanishes for infinitely many natural numbers x ', where $\theta(x)$ is a primitive recursive[5] function.

[5] Primitive recursive functions of natural numbers are defined inductively as follows. Suppose $f(x_1,\dots,x_{n-1})$, $g(x_1,\dots,x_n)$, $h(x_1,\dots,x_{n+1})$ are primitive recursive then $\varphi(x_1,\dots,x_n)$ is primitive recursive if it is defined by one of the sets of equations (a) - (e).

(a) $\varphi(x_1,\dots,x_n) = h(x_1,\dots,x_{m-1}, g(x_1,\dots,x_n), x_{m+1},\dots,x_{n-1}, x_n)$, $(1 \leq m \leq n)$

(b) $\varphi(x_1,\dots,x_n) = f(x_1,\dots,x_{n-1})$

(c) $\varphi(x_1) = a$, where $n = 1$ and a is some particular natural number.

(d) $\varphi(x_1) = x_1 + 1 \quad (n = 1)$

(e) $\varphi(x_1,\dots,x_{n-1}, 0) = f(x_1,\dots,x_{n-1})$
$\varphi(x_1,\dots,x_{n-1}, x_n+1) = h(x_1,\dots,x_n, \varphi(x_1,\dots,x_n))$

The class of primitive recursive function is more restricted than the computable functions, but has the advantage that there is a process whereby one can tell of a set of equations whether if defines a primitive recursive function in the manner described above.

If $\varphi(x_1,\dots,x_n)$ is primitive recursive then $\varphi(x_1,\dots,x_n) = 0$ is described as a primitive recursive relation between $x_1,\dots,x_n$.

We shall say that a problem is number theoretic if it has been shown that any solution of the problem may be put in the form of a proof of one or more number theoretic theorems. More accurately we may say that a class of problems is number theoretic if the solution of any one of them can be transformed (by a uniform process) into the form of proofs of number theoretic theorems.

I shall now draw a few consequences from the definition of 'number theoretic theorems', and in section 5 will try to justify confining our considerations to this type of problem.

An alternative form for number theoretic theorems is 'for each natural number x there exists a natural number y such that $\varphi(x, y)$ vanishes', where $\varphi(x, y)$ is primitive recursive and conversely. In other words, there is a rule whereby given the function $\theta(x)$ we can find a function $\varphi(x, y)$, or given $\varphi(x, y)$ we can find a function $\theta(x)$, so that '$\theta(x)$ vanishes infinitely often' is a necessary and sufficient condition for 'for each x there is y so that $\varphi(x, y) = 0$'. In fact given $\theta(x)$ we define

$$\varphi(x, y) = \theta(y) + \alpha(x, y)$$

where $\alpha(x, y)$ is the (primitive recursive) function with the properties

$$\begin{aligned} \alpha(x, y) &= 1 \quad (y \leq x) \\ &= 0 \quad (y > x) \end{aligned}$$

If on the other hand we are given $\varphi(x, y)$ we define $\theta(x)$ by the equations

$$\begin{aligned} \theta_1(0) &= 3 \\ \theta_1(x+1) &= 3 \cdot \tfrac{2}{3}(\theta_1(x))^{\sigma(\varphi(\varpi_3(\theta_1(x)) - 1,\ \varpi_3(\theta_1(x))))} \\ \theta(x) &= \varphi(\varpi_3(\theta_1(x)) - 1,\ \varpi_2(\theta_1(x))) \end{aligned}$$

where $\varpi_r(x)$ is to be defined so as to mean 'the largest s for which r^s divides x ' and $\frac{2}{3}x$ to be defined primitive recursively so as to have its usual meaning if x is a multiple of 3. The function $\sigma(x)$ is to be defined by the equations $\sigma(0) = 0$, $\sigma(x+1) = 1$. It is easily verified that the functions so defined have the desired properties.

We shall now show that questions as to the truth of statements of form 'does $f(x)$ vanish identically', where $f(x)$ is a computable function, can be reduced to questions as to the truth of number theoretic theorems. It is understood that in each case the rule for the calculation of $f(x)$ is given and that one is satisfied that this rule is valid, i.e. that the machine which should calculate $f(x)$ is circle free (Turing [1], 233). The function $f(x)$ being computable is general recursive in the Herbrand-Gödel sense, and therefore by a general theorem due to Kleene[6] is expressible in the form

[6] Kleene [3], 727. This result is really superfluous for our purpose, as the proof that every computable function is general recursive proceeds by showing that these functions are of form (3.2). (Turing [2], 161).

$$\psi(\epsilon y[\varphi(x,y) = 0]) \tag{3.2}$$

where $\epsilon y[\mathfrak{A}(y)]$ means 'the least y for which $\mathfrak{A}(y)$ is true' and $\psi(y)$ and $\varphi(x,y)$ are primitive recursive functions. Then if we define $\rho(x)$ by the equations (3.1) and

$$\rho(x) = \varphi(\varpi_3(\vartheta_1(x)) - 1, \varpi_2(\vartheta_1(x))) + \psi(\varpi_2(\vartheta_1(x)))$$

it will be seen that $f(x)$ vanishes identically if and only if $\rho(x)$ vanishes for infinitely many values of x.

The converse of this result is not quite true. We cannot say that the question as to the truth of any number theoretic theorem is reducible to a question as to whether a corresponding computable function vanishes identically; we should have rather to say that it is reducible to the problem as to whether a certain machine is circle free and calculates an identically vanishing function. But more is true: every number theoretic theorem is equivalent to the statement that a corresponding machine is circle free. The behavior of the machine may be described roughly as follows: the machine is one for the calculation of the primitive recursive function $\theta(x)$ of the number theoretic problem, except that the results of the calculation are first arranged in a form in which the figures 0 and 1 do not occur, and the machine is then modified so that whenever it has been found that the function vanishes for some value of the argument, then 0 is printed. The machine is circle free if and only if an infinity of these figures are printed, i.e. if and only if $\theta(x)$ vanishes for infinitely many values of the argument. That, on the other hand, questions as to circle freedom may be reduced to questions of the truth of number theoretic theorems follows from the fact that $\theta(x)$ is primitive recursive when it is defined to have the value 0 if a certain machine $\mathcal{M}$ prints 0 or 1 in its $(x+1)$ th complete configuration, and to have the value 1 otherwise.

The conversion calculus provides another normal form for the number theoretic theorems, and the one we shall find the most convenient to use. Every number theoretic theorem is equivalent to a statement of the form ' $\underline{A}(\underline{n})$ is convertible to 2 for every W.F.F. $\underline{n}$ representing a positive integer', $\underline{A}$ being a W.F.F. determined by the theorem; the property of $\underline{A}$ here asserted will be described briefly as '$\underline{A}$ is dual'. Conversely such statements are reducible to number theoretic theorems. The first half of this assertion follows from our results for computable functions, or directly in this way. Since $\theta(x-1)+2$ is primitive recursive it is formally definable, by means of a formula $\underline{G}$ let us say. Now there is (Kleene [1], 232) a W.F.F. $\mathfrak{P}$ with the property that if $\underline{T}(\underline{r})$ is convertible to a formula representing a positive integer for each positive integer r, then $\mathfrak{P}(\underline{T}, \underline{n})$ is convertible to $\underline{s}$ where s is the nth positive integer t (if there is one) for which $\underline{T}(\underline{t})$ conv 2; if $\underline{T}(\underline{t})$ conv 2 for less than n values of t then $\mathfrak{P}(\underline{T}, \underline{n})$ has no normal form. The formula $\underline{G}(\mathfrak{P}(\underline{G}, \underline{n}))$ will therefore be convertible to 2 if and only if $\theta(x)$ vanishes for at least n values of x, and will be convertible to 2 for every positive integer x if and only if $\theta(x)$ vanishes infinitely often. To prove the second half of the assertion we take Gödel representations for the formulae of the conversion calculus. Let $c(x)$ be 0 if x is the G. R. of 2 (i.e. if x is $2^3 \cdot 3^{10} \cdot 5 \cdot 7^3 \cdot 11^{28} \cdot 13 \cdot 17 \cdot 19^{10} \cdot 23^2 \cdot 29 \cdot 31 \cdot 37^{28} \cdot 41^2 \cdot 43 \cdot 47^{28} \cdot 53^2 \cdot 59^2 \cdot 61^2 \cdot 67^2$) and otherwise be 1. Take an enumeration of the G. R. of the formulae into which $\underline{A}(\underline{m})$ is convertible: let $a(m, n)$ be the nth number in the enumeration. We can arrange the enumeration so that $a(m, n)$ is primitive recursive. Now the statement that $\underline{A}(\underline{m})$

is convertible to 2 for every positive integer m is equivalent to the statement that for each positive integer m there is a positive integer n such that $c(a(m,n))=0$, and this is number theoretic.

It is easy to show that a number of unsolved problems such as the problem as to the truth of Fermat's last theorem are number theoretic. There are, however, also problems of analysis which are number theoretic. The Riemann hypothesis gives us an example of this. We denote by $\zeta(s)$ the function defined for $\mathcal{R}s=\sigma>1$ by the series $\sum_{n=1}^{\infty}\frac{1}{n^s}$ and over the rest of the complex plane with the exception of the point $s=1$ by analytic continuation. The Riemann hypothesis asserts that this function does not vanish in the domain $\sigma>\frac{1}{2}$. It is easily shown that this is equivalent to saying that it does not vanish for $2>\sigma>\frac{1}{2}$, $\mathcal{R}s=t\geqslant 2$ i.e. that it does not vanish inside any rectangle $2>\sigma>\frac{1}{2}+\frac{1}{T}$, $T>t>2$ where T is an integer greater than 2. Now the function satisfies the inequalities

$$\left.\begin{array}{l}\left|\zeta(s)-\sum_{1}^{n}n^{-s}-\frac{N^{1-s}}{s-1}\right|<2t(N-2)^{-\frac{1}{2}}\\ \left|\zeta(s)-\zeta(s')\right|<|s-s'|.60t\end{array}\right\}\begin{array}{l}2<\sigma<\frac{1}{2},\ t\geqslant 2\\ 2<\sigma'<\frac{1}{2},\ t'\geqslant 2\end{array}$$

and we can define a primitive recursive function $\xi(l,l',m,m',N,M)$ such that

$$\left|\xi(l,l',m,m',N,M)-M\left|\sum_{1}^{N}n^{-s}+\frac{N^{1-s}}{s-1}\right|\right|<2\quad\left(s=\frac{l}{l'}+i\frac{m}{m'}\right)$$

and therefore if we put

$$\xi(l,M,m,M,M^2+2,M)=X(l,m,M)$$

we shall have

$$\left|\zeta\left(\frac{l+\vartheta}{M}+i\,\frac{m+\vartheta'}{M}\right)\right| \geqslant \frac{X(l,m,M)-122T}{M}$$

$$\frac{1}{2}+\frac{1}{T}\leq\frac{l-1}{M}<\frac{l+1}{M}<2-\frac{1}{M},\ 2<\frac{m-1}{M}<\frac{m+1}{M}<T,\ -1<\vartheta<1,\ -1<\vartheta'<1$$

if we define $B(M,T)$ to be the smallest value of $X(l,m,M)$ for which $\frac{1}{2}+\frac{1}{T}+\frac{1}{M}\leq\frac{l}{M}<2-\frac{1}{M},\ 2+\frac{1}{M}<\frac{m}{M}<T-\frac{1}{M}$, then the Riemann hypothesis is true if for each T there is M satisfying $B(M,T)>122T$. If on the other hand there is T such that for all M, $B(M,T)\leq 122T$, the Riemann hypothesis is false; for let l_M, m_M be such that $X(l_M, m_M, M)\leq 122T$ then $\left|\zeta\left(\frac{l_M+i m_M}{M}\right)\right|\leq\frac{244T}{M}$

Now if a is a condensation point of the sequence $\frac{l_M+i m_M}{M}$ then since $\zeta(s)$ is continuous except at $s=1$ we must have $\zeta(a)=0$ implying the falsity of the Riemann hypothesis. Thus we have reduced the problem to the question as to whether for each T there is M for which $B(M,T)>122T$. $B(M,T)$

is primitive recursive, and the problem is therefore number theoretic.

4. A type of problem which is not number theoretic.[7]

[7] Compare Rosser [1].

Let us suppose that we are supplied with some unspecified means of solving number theoretic problems; a kind of oracle as it were. We will not go any further into the nature of this oracle than to say that it cannot be a machine. With the help of the oracle we could form a new kind of machine (call them o-machines), having as one of its fundamental processes that of solving a given number theoretic problem. More definitely these machines are to behave in this way. The moves of the machine are determined as usual by a table except in the case of moves from a certain internal configuration $\mathfrak{o}$. If the machine is in the internal configuration $\mathfrak{o}$ and if the sequence of symbols marked with $\mathfrak{l}$ is then the well formed[8] formula $\underline{\mathfrak{A}}$, then the machine goes into the internal

[8] Without real loss of generality we may suppose that $\underline{\mathfrak{A}}$ is always well formed.

configuration $\mathfrak{p}$ or $\mathfrak{t}$ according as it is or is not true that $\underline{\mathfrak{A}}$ is dual. The decision as to which is the case is referred to the oracle.

These machines may be described by tables of the same kind as used for the description of a-machines, there being no entries, however, for the internal configuration $\mathfrak{o}$. We obtain description numbers from these tables in the same way as before. If we make the convention that in assigning numbers to internal configurations $\mathfrak{o}$, $\mathfrak{p}$, $\mathfrak{t}$ are always to be q_2, q_3, q_4, then the description numbers determine the behaviour of the machines uniquely.

Given any one of these machines we may ask ourselves the question whether or not it prints an infinity of figures 0 or 1; I assert that this class of problems is not number theoretic. In view of the definition of 'number theoretic problem' this means to say that it is not possible to construct an o-machine which when supplied[9] with the description of any other o-machine will determine whether that machine is o-circle free. The argument may be taken over directly from Turing [1], p. 8. We say that a number is o-satisfactory if it is the description number of an o-circle free machine. Then if there is an o-machine which will determine of any integer whether it is o-satisfactory then there is also an o-machine to calculate the values of the function $1 - \varphi_n(n)$. Let $r(n)$ be the nth o-satisfactory number and let $\varphi_n(m)$ be the mth figure printed by the o-machine whose description number is n. This o-machine is circle free and there is therefore an o-satisfactory number K such that $\varphi_K(n) = 1 - \varphi_n(n)$ all n. Puting $n = K$ yields a contradiction. This completes the proof that problems of circle freedom of o-machines are not number theoretic.

Propositions of the form that an o-machine is o-circle free can always be put in the form of propositions obtained from formulae of the functional calculus of first order by replacing <u>some</u> of the functional variables by primitive recursive relations. Compare footnote[6].

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[9] Compare Turing [1], § 6,7.

5. Syntactical theorems as number theoretic theorems.

I shall mention a property of number theoretic theorems which suggests that there is reason for regarding them as of particular importance.

Suppose that we have some axiomatic system of a purely formal nature. We do not interest ourselves at all in interpretations for the formulae of this system. They are to be regarded as of interest for themselves. An example of what is in mind is afforded by the conversion calculus (§1). Every sequence of symbols '$\underline{A}$ conv $\underline{B}$' where $\underline{A}$ and $\underline{B}$ are well formed formulae, is a formula of the axiomatic system and is provable if the W.F.F. $\underline{A}$ is convertible to $\underline{B}$. The rules of conversion give us the rules of procedure in this axiomatic system.

Now consider a new rule of procedure which is reputed to yield only formulae provable in the original sense. We may ask ourselves whether such a rule is valid. The statement that such a rule is valid would be number theoretic. To prove this let us take Gödel representations for the formulae, and an enumeration of the provable formulae; let $\varphi(r)$ be the G. R. of the r th formula in the enumeration. We may suppose $\varphi(r)$ is primitive recursive if we do not mind repetitions in the enumeration. Let $\psi(r)$ be the G. R. of the r th formula obtained by the new rule, then the statement that this new rule is valid is equivalent to the assertion of

$$(r)(\exists s)\left[\psi(r) = \varphi(s)\right]$$

(the domain of individuals being the natural numbers). It has been shown in § 3 that such statements are number theoretic.

It might plausibly be argued that all theorems of mathematics which have any significance when taken alone, are in effect syntactical theorems of this kind, stating the validity of certain 'derived rules' of procedure. Without going so far as this I should say that theorems of this kind have an importance which makes it worth while to give them special consideration.

6. Logic formulae

We shall call a formula $\underline{L}$ a <u>logic formula</u> (of, if it is clear that we are speaking of a W.F.F., simply a <u>logic</u>) if it has the property that if $\underline{A}$ is a formula such that $\underline{L}(\underline{A})$ conv 2 then $\underline{A}$ is dual.

A logic formula gives us a means of satisfying ourselves of the truth of number theoretic theorems. For to each number theoretic proposition there corresponds a W.F.F. $\underline{A}$ which is dual if and only if the proposition is true. Now if $\underline{L}$ is a logic and $\underline{L}(\underline{A})$ conv 2 then $\underline{A}$ is dual and we know that the corresponding number theoretic proposition is true. It does not follow that if $\underline{L}$ is a logic we can use $\underline{L}$ to satisfy ourselves of the truth of <u>any</u> true number theoretic theorem.

If $\underline{L}$ is a logic the set of formulae $\underline{A}$ for which $\underline{L}(\underline{A})$ conv 2 will be called the <u>extent</u> of $\underline{L}$.

It may be proved by the use of (D), (E) p 7, that there is a formula X such that if $\underline{M}$ has a normal form and no free variables and is not convertible to 2, then $X(\underline{M})$ conv 1, but if $\underline{M}$ conv 2 then $X(\underline{M})$ conv 2. If $\underline{L}$ is a logic then $\lambda x . X(\underline{L}(x))$ is also a logic, whose extent is the same as that of $\underline{L}$, and has the property that if $\underline{A}$ has no free variables then $\{\lambda x . X(\underline{L}(x))\}(\underline{A})$ is always convertible to 1 or to 2 or else has no normal form. A logic with this property will be said to be <u>standardized</u>.

We shall say that a logic $\underline{L}'$ is <u>at least as complete as</u> a logic $\underline{L}$ if the extent of $\underline{L}$ is a subset of the extent of $\underline{L}'$. The logic $\underline{L}'$ will be <u>more complete than</u> $\underline{L}$ if the extent of $\underline{L}$ is a proper subset of the extent of $\underline{L}'$.

Suppose that we have an effective set of rules by which we can prove formulae to be dual; i.e. we have a system of symbolic logic in which the propositions proved are of the form that certain formulae are dual. Then we can find a logic formula whose extent consists of just those formulae which can be proved to be dual by the rules; that is to say that there is a rule for obtaining the logic formula from the system of symbolic logic. In fact the system of symbolic logic enables us to obtain[10] a computable function of posi-

[12] Compare Turing [1], 252, second footnote, [2], 156.

tive integers whose values run through the Gödel representations of the formulae provable by means of the given rules. By the theorem of equivalence of computable and λ-definable functions there is a formula $\underline{J}$ such that $\underline{J}(1), \underline{J}(2), \ldots$ are the G. R. of these formulae. Now let

$$W \to \lambda j v.\, \vartheta(\lambda u.\, \delta(j(u), v), 1, \underline{I}, 2)$$

then I assert that $W(\underline{J})$ is a logic with the required properties. The properties of ϑ imply that $\vartheta(\underline{C}, 1)$ is convertible to the least positive integer $\underline{n}$ for which $\underline{C}(\underline{n})$ conv 2 and has no normal form if there is no such integer. Consequently $\vartheta(\underline{C}, 1, \underline{I}, 2)$ is convertible to 2 if $\underline{C}(\underline{n})$ conv 2 for some positive integer n, and has no normal form otherwise. That is to say that $W(\underline{J}, \underline{A})$ conv 2 if and only if $\delta(\underline{J}(\underline{n}), \underline{A})$ conv 2, some n, i.e. if $\underline{J}(\underline{n})$ conv $\underline{A}$ some n.

There is conversely a formula W' such that if $\underline{L}$ is a logic then $W'(\underline{L})$ enumerates the extent of $\underline{L}$. For there is a formula $\underline{\varphi}$ such that $\underline{\varphi}(\underline{L}, \underline{A}, \underline{n})$ conv 2 if and only if $\underline{L}(\underline{A})$ is convertible to 2 in less than n steps. We then put

$$W' \rightarrow \lambda l n . \, form(\varpi(2, \theta(\lambda x . \varphi(l, form(\varpi(2,x)), \varpi(3,x)), n)))$$

Of course $W'(W(\underline{J}))$ will normally be entirely different from $\underline{J}$ and $W(W'(\underline{L}))$ from $\underline{L}$.

In the case where we have symbolic logic whose propositions can be interpreted as number theoretic theorems, but are not expressed in the form of the duality of formulae we shall again have a corresponding logic formula, but its relation to the symbolic logic will not be so simple. As an example let us take the case that the symbolic logic proves that certain primitive recursive functions vanish infinitely often. As was shown in § 3 we can associate with each such proposition a W.F.F. which is dual if and only if the proposition is true. When we replace the propositions of the symbolic logic by theorems on the duality of formulae in this way our previous argument applies, and we obtain a certain logic formula $\underline{L}$. However, $\underline{L}$ does not determine uniquely which are the propositions provable in the symbolic logic; for it is possible that '$\theta_1(x)$ vanishes infinitely often' and '$\theta_2(x)$ vanishes infinitely often' are both associated with '$\underline{A}$ is dual', and that the first of these propositions is provable in the system, but the second not. However, if we suppose that the system of symbolic logic is sufficiently powerful to be able to carry out the argument on p. 15 then this difficulty cannot arise. There is also the possibility that there may be formulae in the extent of $\underline{L}$ with no propositions of the form '$\theta(x)$ vanishes infinitely often' corresponding to them. But to each such formula we can assign (by a different argument) a proposition $\mathcal{P}$ of the symbolic logic which is the necessary and sufficient condition for $\underline{A}$ to be dual. With $\mathcal{P}$ is associated (in the first way) a formula $\underline{A}'$. Now $\underline{L}$ can always be modified so that its extent contains $\underline{A}'$ whenever it contains $\underline{A}$.

We shall be interested principally in questions of completeness. Let us suppose that we have a class of systems of symbolic logic the propositions of these systems being expressed in a uniform notation and interpretable as number theoretic theorems; suppose also there is a rule by which we can assign to each proposition $\mathcal{P}$ of the notation a W.F.F. $\underline{A}_{\mathcal{P}}$ which is dual if and only if $\mathcal{P}$ is true, and that to each W.F.F. $\underline{A}$ we can assign a proposition $\mathcal{P}_{\underline{A}}$ which is the necessary and sufficient condition for $\underline{A}$ to be dual. $\mathcal{P}_{\underline{A}_{\mathcal{P}}}$ is to be expected to differ from $\mathcal{P}$. To each symbolic logic C we can assign two logic formulae $\underline{L}_C$ and $\underline{L}_C'$. A formula $\underline{A}$ belongs to the extent of $\underline{L}_C$ if $\mathcal{P}_{\underline{A}}$ is provable in C, while the extent of $\underline{L}_C'$ consists of all $\underline{A}_{\mathcal{P}}$ where $\mathcal{P}$ is provable in C. Let us say that the class of symbolic logics is complete if each true proposition is provable in one of them: let us also say that a class of logic formulae is complete if the set theoretic sum of the extents of these logics includes all dual formulae. I assert that a necessary condition for a class of symbolic logics C to be complete is that the class of logics $\underline{L}_C$ be complete, while a sufficient condition is that the class of logics $\underline{L}_C'$ be complete. Let us suppose that the class of symbolic logics is complete; consider $P_{\underline{A}}$ where $\underline{A}$ is arbitrary but dual. It must be provable in one of the systems, C say. $\underline{A}$ therefore belongs to the extent of $\underline{L}_C$, i.e. the class of logics $\underline{L}_C$ is complete. Now suppose the class of logics $\underline{L}_C'$ is complete. Let P be an arbitrary true proposition of the notation; $\underline{A}_P$ must belong to the extent of some $\underline{L}_C'$, and this means that P is provable in C.

We shall say that a single logic formula $\underline{L}$ is complete if its extent includes all dual formulae; that is to say that it is ~~dual~~ complete if it enables us to prove every true number theoretic theorem. It is a consequence of the theorem of Gödel (if suitably extended) that no logic formula is complete, and this also follows from (C) p. 6, or from the results of Turing [1] § 8, when taken in conjunction with § 3 of the present paper. The idea of completeness of a logic formula will not therefore be very important, although it is useful to have a term for it.

Suppose $\underline{Y}$ is a W.F.F. such that $\underline{Y}(\underline{n})$ is a logic for each positive integer $\underline{n}$. The formulae of the extent of $\underline{Y}(\underline{n})$ are enumerated by $W(\underline{Y}(\underline{n}))$, and the combined extents of these logics by $\lambda r.\ W(\underline{Y}(\varpi(2,r)), \varpi(3,r))$. Putting

$$\Gamma \rightarrow \lambda y.\ W'(\lambda r.\ W(y(\varpi(2,r)), \varpi(3,r)))$$

$\Gamma(\underline{Y})$ is a logic whose extent is the combined extent of $\underline{Y}(1)$, $\underline{Y}(2)$, $\underline{Y}(3)$, . . .

To each W.F.F. $\underline{L}$ we can assign a W.F.F. $V(\underline{L})$ such that the necessary and sufficient condition for $\underline{L}$ to be a logic formula is that $V(\underline{L})$ be dual. Let Nm be a W.F.F. which enumerates all formulae with normal forms. Then the condition that $\underline{L}$ be a logic is that $\underline{L}(Nm(\underline{r}), \underline{s})$ conv 2 for all positive integers r, s, i.e. that $\lambda a . \underline{L}(Nm(\varpi(2,a)), \varpi(3,a))$ be dual. We may therefore put

$$V \rightarrow \lambda l a . \, l(Nm(\varpi(2,a)), \varpi(3,a))$$

7. Ordinals.

We begin our treatment of ordinals with some brief definitions from the Cantor theory of ordinals, but for the understanding of some of the proofs a greater amount of the Cantor theory is necessary than is here set out.

Suppose we have a class determined by the propositional function $D(x)$ and a relation $G(x,y)$ ordering them, i.e. satisfying

$$\left.\begin{array}{ll} G(x,y) \,\&\, G(y,z) \supset G(x,z) & \text{i} \\ D(x) \,\&\, D(y) \supset G(x,y) \vee G(y,x) \vee x=y & \text{ii} \\ G(x,y) \supset D(x) \,\&\, D(y) & \text{iii} \\ \sim G(x,x) & \text{iv} \end{array}\right\} \qquad (7.1)$$

The class defined by $D(x)$ is then called a <u>series</u> with the ordering relation $G(x,y)$. The series is said to be <u>well ordered</u> and the ordering relation is called an <u>ordinal</u> if every sub-series which is not void has a first term, i.e. if

$$(D')\Big\{(\exists x)(D'(x)) \,\&\, (x)(D'(x) \supset D(x)) \supset$$
$$\supset (\exists z)(y)\big[D'(z) \,\&\, (D'(x) \supset G(z,y) \vee z=y)\big]\Big\} \qquad (7.2)$$

The condition (7.2) is equivalent to another, more suitable for our purposes, namely the condition that every descending subsequence must terminate; formally

$$(x)\Big\{D'(x) \supset D(x) \,\&\, (\exists y)(D'(y) \,\&\, G(y,x))\Big\} \supset (x)(\sim D'(x)) \qquad (7.3)$$

The ordering relation $G(x,y)$ is said to be similar to $G'(x,y)$ if there is a one-one correspondence between the series transforming the one relation into the other. This is best expressed formally

$$(\exists M)\Big[(x)\big\{D(x) \supset (\exists x')\, M(x,x')\big\} \;\&\; (x')\big\{D'(x') \supset (\exists x)\, M(x,x')\big\}$$
$$\&\;\Big\{\big(M(x,x') \;\&\; M(x,x'')\big) \vee \big(M(x',x) \;\&\; M(x'',x)\big)\Big\} \supset x' = x''\Big\} \qquad (7.4)$$
$$\&\;\Big\{M(x,x') \;\&\; M(y,y') \supset \big(G(x,y) \equiv G'(x',y')\big)\Big\}\Big]$$

Ordering relations are regarded as belonging to the same ordinal if and only if they are similar.

We wish to give names to all the ordinals, but this will not be possible until they have been restricted in some way; the class of ordinals as at present defined is more than enumerable. The restrictions we actually put are these: $D(x)$ is to imply that x is a positive integer; $D(x)$ and $G(x,y)$ are to be computable properties. Both of the propositional functions $D(x)$, $G(x,y)$ can then be described by means of a single W.F.F. $\underline{\Omega}$ with the properties.

$\underline{\Omega}(\underline{m},\underline{n})$ conv 4 unless both $D(m)$ and $D(n)$ are true,

$\underline{\Omega}(\underline{m},\underline{m})$ conv 3 if $D(m)$ is true,

$\underline{\Omega}(\underline{m},\underline{n})$ conv 2 if $D(m)$, $D(n)$, $G(m,n)$, $\sim(m=n)$ are true,

$\underline{\Omega}(\underline{m},\underline{n})$ conv 1 if $D(m)$, $D(n)$, $\sim G(m,n)$, $\sim(m=n)$, are true,

Owing to the conditions to which $D(x)$, $G(x,y)$ are subjected $\underline{\Omega}$ must further satisfy

(a) if $\underline{\Omega}(\underline{m},\underline{n})$ is convertible to 1 or 2 then $\underline{\Omega}(\underline{m},\underline{m})$ and $\underline{\Omega}(\underline{n},\underline{n})$ are convertible to 3,

(b) if $\underline{\Omega}(\underline{m},\underline{m})$ and $\underline{\Omega}(\underline{n},\underline{n})$ are convertible to 3 then $\underline{\Omega}(\underline{m},\underline{n})$ is convertible to 1, 2, or 3,

(c) if $\underline{\Omega}(\underline{m},\underline{n})$ is convertible to 1 then $\underline{\Omega}(\underline{n},\underline{m})$ is convertible to 2 and conversely,

(d) if $\underline{\Omega}(\underline{m},\underline{n})$ and $\underline{\Omega}(\underline{n},\underline{p})$ are convertible to 1 then $\underline{\Omega}(\underline{m},\underline{p})$

is also,

(e) there is no sequence $m_1, m_2, \ldots$ such that $\underline{\Omega}(\underline{m}_{i+1}, \underline{m}_i)$ conv 2 for each positive integer i,

(f) $\underline{\Omega}(\underline{m}, \underline{n})$ is always convertible to 1, 2, 3, or 4.

If a formula $\underline{\Omega}$ satisfies these conditions then there are corresponding propositional functions $D(x)$, $G(x, y)$. We shall therefore say that $\underline{\Omega}$ is an <u>ordinal formula</u> if it satisfies the conditions (a) - (f). It will be seen that a consequence of this definition is that $\underline{Dt}$ is an ordinal formula. It represents the ordinal ω. The definition we have given does not pretend to have virtues such as elegance or convenience. It has been introduced rather to fix our ideas and to show how it is possible in principle to describe ordinals by means of well formed formulae. The definitions could be modified in a number of ways. Some such modifications are quite trivial; they are typified by modifications such as changing the numbers 1,2,3,4, used in the definition to some others. Two such definitions will be said to be equivalent; in general we shall say that two definitions are equivalent if there are W.F.F. $\underline{T}$, $\underline{T}'$ such that if $\underline{A}$ is an ordinal formula under one definition and represents the ordinal α, then $\underline{T}'(\underline{A})$ is an ordinal formula under the second definition and represents the same ordinal, and conversely if $\underline{A}'$ is an ordinal formula under the second definition representing α, then $\underline{T}(\underline{A}')$ represents α under the first definition. Besides definitions equivalent in this sense to our original definition there are a number of other possibilities open.

Suppose for instance that we do not require $\mathfrak{D}(x)$ and $G(x,y)$ to be computable, but only that $\mathfrak{D}(x)$ and $G(x,y) + x < y$ be axiomatic.[13] This leads to a definition of ordinal formula which is (presumably) not equivalent to the definition we are using.[14]

[13] To require $G(x,y)$ to be axiomatic would amount to requiring $G(x,y)$ computable on account of (7.1) ii .

[14] On the other hand if $\mathfrak{D}(x)$ be axiomatic and $G(x,y)$ computable in the modified sense that there is a rule for determining whether $G(x,y)$ is true which leads to a definite result in all cases where $\mathfrak{D}(x)$ and $\mathfrak{D}(y)$ are true, the corresponding definition of ordinal formula is equivalent to our definition. To give the proof would be too much of a digression. Probably a number of other equivalences of this kind hold.

There are numerous possibilities, and little to guide us as to which definition should be chosen. No one of them could well be described as 'wrong'; some of them may be found more valuable in applications than others, and the particular choice we have made has been partly determined by the applications we have in view. In the case of theorems of a negative character one would wish to prove them for each one of the possible definitions of 'ordinal formula'. This program could I think be carried through for the negative results of § 9, 10.

Before leaving the subject of possible ways of defining ordinal formulae I must mention another definition due to Church and Kleene (Church and Kleene [1]). We can make use of this definition in constructing ordinal logics, but it is more convenient to use a slightly different definition which is equivalent (in the sense described on p. 29) to the Church-Kleene definition as modified in Church [4].

Introduce the abbreviations

$$U \rightarrow \lambda u f x . u(\lambda y . f(y(I, x)))$$

$$Suc \rightarrow \lambda a u f x . f(a(u, f, x))$$

We define first a partial ordering relation '$<$' which holds between certain pairs of W.F.F. (conditions (1) - (5)).

(1) If $\underline{A}$ conv $\underline{B}$ then $\underline{A} < \underline{C}$ implies $\underline{B} < \underline{C}$ and $\underline{C} < \underline{A}$ implies $\underline{C} < \underline{B}$.

(2) $\underline{A} < Suc(\underline{A})$

(3) For any positive integers $m, n,$ $\lambda u f x . \underline{R}(\underline{n}) < \lambda u f x . \underline{R}(\underline{m})$ implies $\lambda u f x . \underline{R}(\underline{n}) < \lambda u f x . u(\underline{R})$.

(4) If $\underline{A} < \underline{B}$ and $\underline{B} < \underline{C}$ then $\underline{A} < \underline{C}$. (1) - (4) are required for any W.F.F. $\underline{A}, \underline{B}, \underline{C}, \lambda u f x . \underline{R}$.

(5) The relation $\underline{A} < \underline{B}$ holds only when compelled to do so by (1) - (4).

We define C-K ordinal formulae by the conditions (6) - (10).

(6) If $\underline{A}$ conv $\underline{B}$ and $\underline{A}$ is a C-K ordinal formula then $\underline{B}$ is a C-K ordinal formula.

(7) U is a C-K ordinal formula.

(8) If $\underline{A}$ is a C-K ordinal formula then $Suc(\underline{A})$ is a C-K ordinal formula.

(9) If $\lambda u f x . \underline{R}(\underline{n})$ is a C-K ordinal formula and $\lambda u f x . \underline{R}(\underline{n}) < < \lambda u f x . \underline{R}(S(\underline{n}))$ for each positive integer n then $\lambda u f x . u(\underline{R})$ is a C-K ordinal formula.

(10) A formula is a C-K ordinal formula only if compelled to be so by (6) - (9).

The representation of ordinals by formulae is described by (11) - (15).

(11) If $\underline{A}$ conv $\underline{B}$ and $\underline{A}$ represents α then $\underline{B}$ represents α.

(12) U represents 1.

(13) If $\underline{A}$ represents α then $Suc(\underline{A})$ represents $\alpha+1$.

(14) If $\lambda u f x.\, \underline{R}(\underline{n})$ represents α_n for each positive integer n then $\lambda u f x.\, u(\underline{R})$, represents the upper bound of the sequence $\alpha_1, \alpha_2, \alpha_3 \ldots$.

(15) A formula represents an ordinal only when compelled to do so by (11) - (14).

We denote any ordinal represented by $\underline{A}$ by $\Xi_{\underline{A}}$ without prejudice to the possibility that more than one ordinal may be represented by $\underline{A}$. We shall write $\underline{A} \leq \underline{B}$ to mean $\underline{A} < \underline{B}$ or $\underline{A}$ conv $\underline{B}$.

In proving properties of C-K ordinal formulae we shall often use a kind of analogue of the principle of transfinite induction. If φ is some property and we have

(a) If $\underline{A}$ conv $\underline{B}$ and $\varphi(\underline{A})$ then $\varphi(\underline{B})$.

(b) $\varphi(U)$.

(c) If $\varphi(\underline{A})$ then $\varphi(Suc(\underline{A}))$.

(d) If $\varphi(\lambda u f x.\, \underline{R}(\underline{n}))$ and $\lambda u f x.\, \underline{R}(\underline{n}) < \lambda u f x.\, \underline{R}(S(\underline{n}))$ for each positive integer n then $\varphi(\lambda u f x.\, u(\underline{R}))$

then $\varphi(\underline{A})$ for each C-K ordinal formula $\underline{A}$. To prove the validity of this principle we have only to observe that the class of formulae $\underline{A}$ satisfying $\varphi(\underline{A})$ is one of those of which the class of C-K ordinal formulae was defined to be the smallest. We can use this principle to help us prove:-

(i) Every C-K ordinal formula is convertible to the form $\lambda uf x.\underline{B}$ where $\underline{B}$ is in normal form.

(ii) There is a method by which one can determine of any C-K ordinal formula into which of the forms U, $Suc(\lambda uf x.\underline{B})$, $\lambda uf x.u(\underline{R})$, where u is free in $\underline{R}$, it is convertible, and to determine $\underline{B}$, $\underline{R}$. In each case $\underline{B}$, $\underline{R}$ are unique apart from conversions.

(iii) If $\underline{A}$ represents any ordinal $\Xi_{\underline{A}}$ is unique. If $\Xi_{\underline{A}}$, $\Xi_{\underline{B}}$ exist and $\underline{A} < \underline{B}$ then $\Xi_{\underline{A}} < \Xi_{\underline{B}}$.

(iv) If $\underline{A}$, $\underline{B}$, $\underline{C}$ are C-K ordinal formulae and $\underline{B} < \underline{A}$, $\underline{C} < \underline{A}$ then either $< \underline{B}$, $\underline{B} < \underline{C}$ or $\underline{B}$ conv $\underline{C}$.

(v) A formula $\underline{A}$ is a C-K ordinal formula if

(A) $U \leqslant \underline{A}$

(B) If $\lambda uf x.u(\underline{R}) \leqslant \underline{A}$ and n is a positive integer, then $\lambda uf x.\underline{R}(\underline{n}) < \lambda uf x.\underline{R}(S(\underline{n}))$.

(C) For any two W.F.F. $\underline{B}$, $\underline{C}$ with $\underline{B} < \underline{A}$, $\underline{C} < \underline{A}$ we have $\underline{B} < \underline{C}$, $\underline{C} < \underline{B}$ or $\underline{B}$ conv $\underline{C}$, but never $\underline{B} < \underline{B}$.

(D) There is no infinite sequence $\underline{B}_1$, $\underline{B}_2$, . . . for which $\underline{B}_r < \underline{B}_{r-1} < \underline{A}$ each r.

(vi) There is a formula H such that if $\underline{A}$ is a C-K ordinal formula then $H(\underline{A})$ is an ordinal formula representing the same ordinal. $H(\underline{A})$ is not an ordinal formula unless $\underline{A}$ is a C-K ordinal formula.

<u>Proof of (i)</u>. Take $\varphi(\underline{A})$ to be '$\underline{A}$ is convertible to the form

$\lambda u f x . \underline{B}$ where $\underline{B}$ is in normal form'. The conditions (a), (b) are trivial. For (c) suppose $\underline{A}$ conv $\lambda u f x . \underline{B}$ where $\underline{B}$ is in normal form, then $Suc(\underline{A})$ conv $\lambda u f x . f(\underline{B})$ and $f(\underline{B})$ is in normal form. For (d) we have only to show that $u(\underline{R})$ has a normal form, i.e. that $\underline{R}$ has a normal form, which is true since $\underline{R}(\underline{I})$ has a normal form.

<u>Proof of (ii)</u>. Since by hypothesis the formula is a C-K ordinal formula we have only to perform conversions on it until it is in one of the forms described. It is not possible to convert it into two of these three forms. For suppose $\lambda u f x . f(\underline{A}(u, f, x))$ conv $\lambda u f x . u(\underline{R})$ and is a C-K ordinal formula; it is therefore convertible to the form $\lambda u f x . \underline{B}$ where $\underline{B}$ is in normal form. But the normal form of $\lambda u f x . u(\underline{R})$ can be obtained by conversions on $\underline{R}$, and that of $\lambda u f x . f(\underline{A}(u, f, x))$ by conversions on $\underline{A}(u, f, x)$ (as follows from Church and Rosser [1] theorem 2) but this would imply that the formula in question had two normal forms, one of form $\lambda u f x . u(\underline{S})$ and one of form $\lambda u f x . f(\underline{C})$, which is impossible. Or suppose U conv $\lambda u f x . u(\underline{R})$ where $\underline{R}$ is a well formed formula with u as a free variable. We may suppose $\underline{R}$ is in normal form. Now U is $\lambda u f x . u(\lambda y . f(y(\underline{I}, x)))$. By (A) p. 6. $\underline{R}$ is identical with $\lambda y . f(y(\underline{I}, x))$ which does not have u as a free variable. It now only remains to show that if $Suc(\lambda u f x . \underline{B})$ conv $Suc(\lambda u f x . \underline{B}')$ and $\lambda u f x . u(\underline{R})$ conv $\lambda u f x . u(\underline{R}')$ then $\underline{B}$ conv $\underline{B}'$ and $\underline{R}$ conv $\underline{R}'$.

If $Suc(\lambda u f x . \underline{B})$ conv $Suc(\lambda u f x . \underline{B}')$ then $\lambda u f x . f(\underline{B})$ conv $\lambda u f x . f(\underline{B}')$

but both of these formulae can be brought to normal form by conversions on $\underline{B}$, $\underline{B}'$ and therefore $\underline{B}$ conv $\underline{B}'$. The same argument applies in the case that $\lambda u f x . u(\underline{R})$ conv $\lambda u f x . u(\underline{R}')$.

<u>Proof of (iii)</u>. To prove the first half take $\varphi(\underline{A})$ to be '$\equiv_{\underline{A}}$ is unique'. (7.5) (a) is trivial and (b) follows from the fact that U is not convertible either to the form $Suc(\underline{A})$ or to $\lambda u f x . u(\underline{R})$ where $\underline{R}$ has u as a free variable. For (c): $Suc(\underline{A})$ is not convertible to the form $\lambda u f x . u(\underline{R})$; the possibility of $Suc(\underline{A})$ representing an ordinal on account of (12) or (14) is therefore eliminated. By (13) $Suc(\underline{A})$ represents $\alpha' + 1$ if $\underline{A}'$ represents α' and $Suc(\underline{A})$ conv $Suc(\underline{A}')$. If we suppose $\underline{A}$ represents α , then $\underline{A}$, $\underline{A}'$ being C-K ordinal formulae are convertible to the forms $\lambda u f x . \underline{B}'$, $\lambda u f x . \underline{B}'$ but then by (ii) $\underline{B}$ conv $\underline{B}'$ i.e. $\underline{A}$ conv $\underline{A}'$, and therefore by the hypothesis $\varphi(\underline{A})$, $\alpha = \alpha'$. Then $\equiv_{Suc(\underline{A})} = \alpha' + 1$ is unique. For (d): $\lambda u f x . u(\underline{R})$ is not convertible to the form $Suc(\underline{A})$ or to U if $\underline{R}$ has u as a free variable. If $\lambda u f x . u(\underline{R})$ represents an ordinal it is therefore in virtue of (14), possibly together with (11). Now if $\lambda u f x . u(\underline{R})$ conv $\lambda u f x . u(\underline{R}')$ then $\underline{R}$ conv $\underline{R}'$, so that the sequence $\lambda u f x . \underline{R}(1)$, $\lambda u f x . \underline{R}(2)$, . . . in (14) is unique apart from conversions. Then by the induction hypothesis the sequence α_1 , α_2, α_3 , . . . is unique. The only ordinal that is represented by $\lambda u f x . u(\underline{R})$ is the upper bound of this sequence which is unique.

For the second half we use a type of argument rather different from our transfinite induction principle. The formulae $\underline{B}$ for which $\underline{A} < \underline{B}$ form the smallest class for which

$Suc(\underline{A})$ belongs to the class.

If $\underline{C}$ belongs to the class then $Suc(\underline{C})$ belongs to it.

If $\lambda u f x . \underline{R}(\underline{n})$ belongs to the class and $\lambda u f x . \underline{R}(\underline{n}) < \lambda u f x . \underline{R}(\underline{m})$ where m, n are some positive integers then $\lambda u f x . u(\underline{R})$ belongs to it.

If $\underline{C}$ belongs to the class and $\underline{C}$ conv $\underline{C}'$ then $\underline{C}'$ belongs to it.

It will suffice to prove that the class of formulae $\underline{B}$ for which either $\Xi_{\underline{B}}$ does not exist or $\Xi_{\underline{A}} < \Xi_{\underline{B}}$ satisfies the conditions (7.6). Now

$$\Xi_{Suc(\underline{A})} = \Xi_{\underline{A}} + 1 > \Xi_{\underline{A}}$$

$$\Xi_{Suc(\underline{C})} > \Xi_{\underline{C}} > \Xi_{\underline{A}} \quad \text{if } \underline{C} \text{ is in the class.}$$

If $\Xi_{\lambda u f x . \underline{R}(\underline{n})}$ does not exist then $\Xi_{\lambda u f x . u(\underline{R})}$ does not exist, and therefore $\lambda u f x . u(\underline{R})$ is in the class. If $\Xi_{\lambda u f x . \underline{R}(\underline{n})}$ exists and is greater than $\Xi_{\underline{A}}$ and $\lambda u f x . \underline{R}(\underline{n}) < \lambda u f x . \underline{R}(\underline{m})$ then

$$\Xi_{\lambda u f x . u(\underline{R})} \geqslant \Xi_{\lambda u f x . \underline{R}(\underline{n})} > \Xi_{\underline{A}}$$

so that $\lambda u f x . u(\underline{R})$ belongs to the class.

<u>Proof of (iv)</u>. We prove this by induction with respect to $\underline{A}$. Take $\varphi(\underline{A})$ to be 'whenever $\underline{B} < \underline{A}$ and $\underline{C} < \underline{A}$ then $\underline{B} < \underline{C}$ or $\underline{C} < \underline{B}$ or $\underline{B}$ conv $\underline{C}$'. $\varphi(U)$ follows from the fact that we never have $\underline{B} < U$. If we have $\varphi(\underline{A})$ and $\underline{B} < Suc(\underline{A})$ then either $\underline{B} < \underline{A}$ or $\underline{B}$ conv $\underline{A}$; for we can find $\underline{D}$ so that $\underline{B} \leq \underline{D}$,

and $\underline{D} < Suc(\underline{A})$ can be proved without appealing either to (1) or (5); (4) does not apply so we must have $\underline{D}$ conv $\underline{A}$. Then if $\underline{B} < Suc(\underline{A})$ and $\underline{C} < Suc(\underline{A})$ we have four possibilities

$$\begin{array}{llll} \underline{B} \text{ conv } \underline{A}, & \underline{C} \text{ conv } \underline{A} \\ \underline{B} \text{ conv } \underline{A}, & \underline{C} < \underline{A} \\ \underline{B} < \underline{A}, & \underline{C} \text{ conv } \underline{A} \\ \underline{B} < \underline{A}, & \underline{C} < \underline{A} \end{array}$$

In the first case $\underline{B}$ conv $\underline{C}$, in the second $\underline{C} < \underline{B}$, in the third $\underline{B} < \underline{C}$ and in the fourth the induction hypothesis applies.

Now suppose that $\lambda uf x . \underline{R}(\underline{n})$ is a C-K ordinal formula, $\lambda uf x . \underline{R}(\underline{n}) < \lambda uf x . \underline{R}(S(\underline{n}))$ and $\varphi(\underline{R}(\underline{n}))$, for each positive integer n, and $\underline{A}$ conv $\lambda uf x . u(\underline{R})$. Then if $\underline{B} < \underline{A}$ this means that $\underline{B} < \lambda uf x . \underline{R}(\underline{n})$ for some n; if we have also $\underline{C} < \underline{A}$ then $\underline{B} < \lambda uf x . \underline{R}(\underline{n}')$, $\underline{C} < \lambda uf x . \underline{R}(\underline{n}')$ some n'. Thus for these $\underline{B}$, $\underline{C}$ the required result follows from $\varphi(\lambda uf x . \underline{R}(\underline{n}'))$.

Proof of (v). The conditions (C), (D) imply that the classes of interconvertible formulae $\underline{B}$, $\underline{B} < \underline{A}$ are well-ordered by the relation '$<$'. We prove (v) by (ordinary) transfinite induction with respect to the order type α of the series formed by these classes; (α is in fact the solution of the equation $1 + \alpha = \Xi_{\underline{A}}$ but we do not need this). We suppose then that (v) is true for all order types less than α. If $\underline{E} < \underline{A}$ then $\underline{E}$ satisfies the conditions of (v) and the corresponding order type is smaller: $\underline{E}$ is therefore a C-K ordinal formula. This expresses all consequences of the induction hypothesis that we need. There are three cases to consider.

(x) $\alpha = 0$

(y) $\alpha = \beta + 1$

(z) α is of neither of the forms (x), (y).

In case (x) we must have $\underline{A}$ conv U on account of (A). In case (y) there is a formula $\underline{D}$ such that $\underline{D} < \underline{A}$, and $\underline{B} \leq \underline{D}$ whenever $\underline{B} < \underline{A}$. The relation $\underline{D} < \underline{A}$ must hold in virtue either of (1), (2), (3), or (4). It cannot be in virtue of (4) for then there would be $\underline{B}$, $\underline{B} < \underline{A}$, $\underline{D} < \underline{B}$ contrary to (C) taken in conjunction with the definition of $\underline{D}$. If it is in virtue of (3) then α is the upper bound of a sequence $\alpha_1, \alpha_2 \ldots$ of ordinals, which are increasing on account of (ii) and the conditions $\lambda u f x . \underline{R}(\underline{n}) < \lambda u f x . \underline{R}(S(y))$ in (3). This is inconsistent with $\alpha = \beta + 1$. This means that (2) applies (after we have eliminated (1) by suitable conversions on $\underline{A}$, $\underline{D}$) and we see that $\underline{A}$ conv $Suc(\underline{D})$; but since $\underline{D} < \underline{A}$, $\underline{D}$ is a C-K ordinal formula, and $\underline{A}$ must therefore be a C-K ordinal formula by (B). Now take case (z). It is impossible that $\underline{A}$ be of form $Suc(\underline{D})$, for then we should have $\underline{B} < \underline{D}$ whenever $\underline{B} < \underline{A}$ which would mean that we had case (y). Since $U < \underline{A}$ there must be an $\underline{F}$ such that $\underline{F} < \underline{A}$ is demonstrable either by (2) or by (3) (after a possible conversion on $\underline{A}$); it must of course be demonstrable by (3). Then $\underline{A}$ is of form $\lambda u f x . u(\underline{R})$. By (3), (B) we see that $\lambda u f x . \underline{R}(\underline{n}) < \underline{A}$ for each positive integer n; each $\lambda u f x . \underline{R}(\underline{n})$ is therefore a C-K ordinal formula. Applying (9), (B) we see that $\underline{A}$ is a C-K ordinal formula.

<u>Proof of vi</u>. To prove the first half it suffices to find a method whereby from a C-K ordinal formula $\underline{A}$ we can find the

corresponding ordinal formula $\underline{\Omega}$. For then there is a formula H_1 such that $H_1(\underline{a})$ conv $\underline{P}$ if a is the G.R. of $\underline{A}$ and P that of $\underline{\Omega}$. H is then to be defined by

$$H \to \lambda a.\, \mathrm{form}(H_1(Gr(a)))$$

The method for finding $\underline{\Omega}$ may be replaced by a method of finding $\underline{\Omega}(\underline{m}, \underline{n})$ given $\underline{A}$ and any two positive integers m, n. We shall arrange the method so that whenever $\underline{A}$ is not an ordinal formula either the calculation of the values does not comes to an end or else the values are not consistent with $\underline{\Omega}$ being an ordinal formula. In this way we can prove the second half of (vi).

Let Ls be a formula such that $Ls(\underline{A})$ enumerates the classes of formulae $\underline{B}$, $\underline{B} < \underline{A}$ (i.e. if $\underline{B} < \underline{A}$ there is one and only one positive integer n for which $Ls(\underline{A}, \underline{n})$ conv $\underline{B}$). Then the rule for finding the value of $\underline{\Omega}(\underline{m}, \underline{n})$ is as follows:-

First determine whether $U \leqslant \underline{A}$ and whether $\underline{A}$ is convertible to the form $\underline{r}(Suc, U)$. This comes to an end if $\underline{A}$ is a C-K ordinal formula.

If $\underline{A}$ conv $\underline{r}(Suc, U)$ and either $m > r+1$ or $n > r+1$ then the value is 4. If $m < n \leqslant r+1$ the value is 2. If $n < m \leqslant r+1$ the value is 1. If $m = n \leqslant r+1$ the value is 3.

If $\underline{A}$ is not convertible to this form we determine whether either $\underline{A}$ or $Ls(\underline{A}, \underline{m})$ is convertible to the form $\lambda u f x.\, u(\underline{R})$ and if either of them is we verify that $\lambda u f x.\, \underline{R}(\underline{u}) < \lambda u f x.\, \underline{R}(S(\underline{u}))$ We shall eventually comes to an affirmative answer if $\underline{A}$ is a C-K ordinal formula.

Having checked this we determine of m, n whether $Ls(\underline{A}, \underline{m}) < Ls(\underline{A}, \underline{n})$, $Ls(\underline{A}, \underline{n}) < Ls(\underline{A}, \underline{m})$, or $m = n$, and the value is to be accordingly 1, 2, or 3.

If $\underline{A}$ is a C-K ordinal formula this process certainly comes to an end. To see that the values so calculated correspond to an ordinal formula, and one representing $\Xi_{\underline{A}}$, first observe that this is so when $\Xi_{\underline{A}}$ is finite. In the other case (iii), (iv) show that $\Xi_{\underline{B}}$ determines a one-one correspondence between the ordinals β, $1 \leq \beta \leq \Xi_{\underline{A}}$ and the classes of interconvertible formulae $\underline{B}$, $\underline{B} < \underline{A}$. If we take $G(m, n)$ to be $Ls(\underline{A}, \underline{m}) < Ls(\underline{A}, \underline{n})$ we see that $G(m, n)$ is the ordering relation of a series of order type[15] $\Xi_{\underline{A}}$ and on the other hand that the values of $\underline{\Omega}(\underline{m}, \underline{n})$ are related to $G(m, n)$ as on p. 29.

To prove the second half suppose $\underline{A}$ is not a C-K ordinal formula. Then one of the conditions (A)-(D) in (v) must not be satisfied. If (A) is not satisfied we shall not obtain a result even in the calculation of $\underline{\Omega}(\underline{1}, \underline{1})$. If (B) is not satisfied, for some positive integers p, q we shall have $Ls(\underline{A}, \underline{p})$ conv $\lambda ufx . u(\underline{R})$ but not $\lambda ufx . \underline{R}(\underline{q}) < \lambda ufx . \underline{R}(S(\underline{q}))$. Then the process of calculating $\underline{\Omega}(\underline{p}, \underline{q})$ will not come to an end. In case of failure of (C) or (D) the values of $\underline{\Omega}(\underline{m}, \underline{n})$ may all be calculable but condition (b), (d), or (e) p. 29, 30 will be violated. Thus if $\underline{A}$ is not a C-K ordinal formula then $H(\underline{A})$ is not an or-

[15] The order type is β where $1 + \beta = \Xi_{\underline{A}}$ but $\beta = \Xi_{\underline{A}}$ since $\Xi_{\underline{A}}$ is infinite.

dinal formula.

I propose now to define three formulae Sum, Lim, Inf of importance in connection with ordinal formulae. As they are comparatively simple they will for once be given almost in full:
The formula Ug is one with the property that $Ug(\underline{m})$ is convertible to the formula representing the largest odd integer dividing m : it is not given in full. P is the predecessor function, $P(S(\underline{m}))$ conv $\underline{m}$.

$$\mathrm{Al} \rightarrow \lambda pxy \,.\, p(\lambda guv \,.\, g(v,u), \lambda uv \,.\, u(I,v), x, y)$$

$$\mathrm{Hf} \rightarrow \lambda m \,.\, P(m(\lambda guv \,.\, g(v, S(u)), \lambda uv \,.\, v(I,u), \underline{1}, \underline{2}))$$

$$\mathrm{Bd} \rightarrow \lambda \omega \omega' a a' x \,.\, \mathrm{Al}(\lambda f \,.\, \omega(a, a, \omega'(a', a', f)), x, \underline{4})$$

$$\begin{aligned}\mathrm{Sum} \rightarrow \lambda \omega \omega' pq \,.\, \mathrm{Bd}(\omega, \omega', \mathrm{Hf}(p), \mathrm{Hf}(q), \mathrm{Al}(p, \mathrm{Al}(q, \omega'(\mathrm{Hf}(p), \mathrm{Hf}(q)), \\ \underline{1}), \mathrm{Al}(q, \underline{2}, \omega(\mathrm{Hf}(p), \mathrm{Hf}(q)))))\end{aligned}$$

$$\begin{aligned}\mathrm{Lim} \rightarrow \lambda zpq \,.\, \{\lambda ab \,.\, \mathrm{Bd}(z(a), z(b), Ug(p), Ug(q), \mathrm{Al}(\mathrm{Dt}(a,b) + \\ + \mathrm{Dt}(b,a), \mathrm{Dt}(a,b), z(a, Ug(p), Ug(q)))))\}(\mathrm{Dt}(\underline{2}, p), \mathrm{Dt}(\underline{2}, q))\end{aligned}$$

$$\mathrm{Inf} \rightarrow \lambda \omega a pq \,.\, \mathrm{Al}(\lambda f \,.\, \omega(a, p, \omega(a, q, f)), \omega(p, q), \underline{4})$$

The essential properties of these formulae are described by

$\mathrm{Al}(2\underline{r} - \underline{1}, \underline{m}, \underline{n})$ conv $\underline{m}$ $\qquad$ $\mathrm{Al}(2\underline{r}, \underline{m}, \underline{n})$ conv $\underline{n}$

$\mathrm{Hf}(2\underline{m})$ conv $\underline{m}$ $\qquad$ $\mathrm{Hf}(2\underline{m} - \underline{1})$ conv $\underline{m}$

$\mathrm{Bd}(\underline{\Omega}, \underline{\Omega}', \underline{a}, \underline{a}', x)$ conv 4 unless both $\underline{\Omega}(\underline{a}, \underline{a})$ conv 3 and $\underline{\Omega}'(\underline{a}', \underline{a}')$ conv 3 in which case it is convertible to x.

If $\underline{\Omega}$, $\underline{\Omega}'$ are ordinal formulae representing α, β respectively then $Sum(\underline{\Omega}, \underline{\Omega}')$ is an ordinal formula representing $\alpha + \beta$. If $\underline{Z}$ is a W.F.F. enumerating a sequence of ordinal formulae representing $\alpha_1, \alpha_2, \ldots$, then $Lim(\underline{Z})$ is an ordinal formula representing the infinite sum $\alpha_1 + \alpha_2 + \alpha_3 + \ldots$. If $\underline{\Omega}$ is an ordinal formula representing α then $Inf(\underline{\Omega})$ enumerates a sequence of ordinal formulae representing all the ordinals less than α without repetitions.

To prove that there is no general method for determining of a formula whether it is an ordinal formula we use an argument akin to that leading to the Burali-Forti paradox, but the emphasis and the conclusion are different. Let us suppose that such an algorithm is available. This enables us to obtain a recursive enumeration $\underline{\Omega}_1, \underline{\Omega}_2, \ldots$ of the ordinal formulae in normal form. There is a formula $\underline{Z}$ such that $\underline{Z}(\underline{n})$ conv $\underline{\Omega}_n$. Now $Lim(\underline{Z})$ represents an ordinal greater than any represented by an $\underline{\Omega}_n$, and has therefore been omitted from the enumeration.

This argument proves more than was originally asserted. In fact it proves that if we take any class E of ordinal formulae in normal form, such that if $\underline{A}$ is any ordinal formula then there is a formula in E representing the same ordinal as $\underline{A}$, then there is no method whereby one can tell whether a W.F.F. ~~no~~ in normal form belongs to E.

8. Ordinal logics.

An ordinal logic is a W.F.F. $\underset{\sim}{\Lambda}$ such that $\underset{\sim}{\Lambda}(\underset{\sim}{\Omega})$ is a logic formula whenever $\underset{\sim}{\Omega}$ is an ordinal formula.

This definition is intended to bring under one heading a number of ways of constructing logics which have recently been proposed or are suggested by recent advances. In this section I propose to show how to obtain some of these ordinal logics.

Suppose we have a class W of logical systems. The symbols used in each of these systems are the same, and a class of sequences of symbols called 'formulae' is defined, independently of the particular system in W. The rules of procedure of a system C define an axiomatic subset of the formulae, they are to be described as the 'provable formulae of C'. Suppose further that we have a method whereby, from any system of C of W we can obtain a new system C', also in W, and such that the set of provable formulae of C' include the provable formulae of C (we shall be most interested in the case where they are included as a proper subset.) It is to be understood that this 'method' is an effective procedure for obtaining the rules of procedure of C' from those of C.

Suppose that to certain of the formulae of W we make correspond number theoretic theorems: by modifying the definition of formula we may suppose that this is done for all formulae. We shall say that one of the systems C is <u>valid</u> if the provability of a formula in C implies the truth of the corresponding number theoretic theorem. Now let the relation of C' to C be such that the validity of C implies the validity of C', and let there be a valid system C_0 in W. Finally suppose that given any computable sequence $C_1, C_2, \ldots$ of systems in W the 'limit system' in which a formula is provable if and only if it is provable in one of the systems C_j also belongs to W. These limit systems are to be regarded, not as functions of the sequence given in extension, but as functions of the rules of formation of their terms. A sequence given in extension may be described by various rules of formation, and there will be several corresponding limit systems. Each of these may be described as <u>a</u> limit system of the sequence.

Under these circumstances we may construct an ordinal logic. Let us associate positive integers with the systems, in such a way that to each C corresponds a positive integer m_C, and m_C completely describes the rules of procedure of C. Then there is a W.F.F. $\underline{K}$, such that $\underline{K}(\underline{m}_C)$ conv $\underline{m}_{C'}$ for each C in W, and there is a W.F.F. $\underline{\Theta}$ such that if $\underline{D}(\underline{r})$ conv $\underline{m}_{C_r}$ for each positive integer r then $\underline{\Theta}(\underline{D})$ conv $\underline{m}_C$ where C is a limit system of $C_1, C_2, \ldots$. With each system C of W it is possible to associate a logic formula $\underline{L}_C$: the relation between them is that if G is a formula of W and the number theoretic theorem corresponding to G (assumed expressed in the conversion calculus form) asserts that $\underline{B}$ is dual, then $\underline{L}_C(\underline{B})$ conv 2 if and only if G is provable in C. There will be a W.F.F. $\underline{G}$ such that $\underline{G}(\underline{m}_C)$ conv $\underline{L}_C$ for each C of W. Put

$$\underline{N} \to \lambda a \,.\, \underline{G}(a(\underline{\Theta}, \underline{K}, \underline{m}_{C_0}))$$

I assert that $\underline{N}(\underline{A})$ is a logic formula for each C-K ordinal formula $\underline{A}$, and that if $\underline{A} < \underline{B}$ then $N(\underline{B})$ is more complete than $N(\underline{A})$, provided that there are formulae provable in C' but not in C for each valid C of W.

To prove this we shall show that to each C-K ordinal formula there corresponds a unique system $C[\underline{A}]$ such that

(i) $\underline{A}(\underline{\Theta}, \underline{K}, \underline{m}_{C_0})$ conv $\underline{m}_{C_0'}$

and that it further satisfies

(ii) $C[U]$ is a limit system of $C_0', C_0', \ldots$

(iii) $C[\mathrm{Suc}(\underline{A})]$ is $(C[\underline{A}])'$

(iv) $C[\lambda u f x . u(\underline{R})]$ is a limit system of $C[\lambda u f x . \underline{R}(\underline{1})]$, $C[\lambda u f x . \underline{R}(\underline{2})]$, . . . ,

$\underline{A}$ and $\lambda u f x . u(\underline{R})$ being assumed to be C-K ordinal formulae. The uniqueness of the system follows from the fact that $\underline{m}_C$ determines C completely. Let us try to prove the existence of $C[\underline{A}]$ for each C-K ordinal formula $\underline{A}$. As we have seen (p.33) it suffices to prove

(a) $C[U]$ exists,

(b) if $C[\underline{A}]$ exists then $C[\mathrm{Suc}(\underline{A})]$ exists,

(c) if $C[\lambda u f x . \underline{R}(\underline{1})]$, $C[\lambda u f x . \underline{R}(\underline{2})]$, . . . exist then $C[\lambda u f x . u(\underline{R})]$ exists.

<u>Proof of (a).</u>

$$\{\lambda y . \underline{K}(y(\underline{I}, \underline{m}_{C_0}))\}(\underline{n}) \quad \text{conv} \quad \underline{K}(\underline{m}_{C_0}) \quad \text{conv} \quad \underline{m}_{C_0'}$$

for all positive integers n, and therefore by the definition of Θ there is a system, which we will call $C[U]$, and which is a limit system of C_0', C_0', . . . , satisfying

$$\underline{\Theta}(\lambda y.\,\underline{K}(y(\underline{I},\underline{m}_{C_0})))\ \textbf{conv}\ \ \underline{m}_{C[U]}$$

But on the other hand

$$U(\underline{\Theta},\underline{K},\underline{m}_{C_0})\ \ \textbf{conv}\ \ \underline{\Theta}(\lambda y.\,\underline{K}(y(\underline{I},\underline{m}_{C_0})))$$

This proves (a) and incidentally (ii)

<u>Proof of (b)</u>.

$$\begin{aligned} Suc(\underline{A},\underline{\Theta},\underline{K},\underline{m}_{C_0})\ &\textbf{conv}\ \ \underline{K}(\underline{A}(\underline{\Theta},\underline{K},\underline{m}_{C_0})) \\ &\textbf{conv}\ \ \underline{K}(\underline{m}_{C[\underline{A}]}) \\ &\textbf{conv}\ \ \underline{m}_{(C[\underline{A}])'} \end{aligned}$$

Hence $C[Suc(\underline{A})]$ exists and is given by (iii).

<u>Proof of (c)</u>.

$$\begin{aligned} \{\{\lambda u f x.\,\underline{R}\}(\underline{\Theta},\underline{K},\underline{m}_{C_0})\}(\underline{u})\ &\textbf{conv}\ \ \{\lambda u f x.\,\underline{R}(\underline{u})\}(\underline{\Theta},\underline{K},\underline{m}_{C_0}) \\ &\textbf{conv}\ \ \underline{m}_{C[\lambda u f x.\,\underline{R}(\underline{u})]} \end{aligned}$$

by hypothesis. Consequently by the definition of $\underline{\Theta}$ there exists C which is a limit system of $C[\lambda u f x.\,\underline{R}(1)], C[\lambda u f x.\,\underline{R}(2)], \ldots$ and satisfies

$$\underline{\Theta}(\{\lambda u f x.\,\underline{R}\}(\underline{\Theta},\underline{K},\underline{m}_{C_0}))\ \textbf{conv}\ \ \underline{m}_C$$

We define $C[\lambda u f x.\,u(\underline{R})]$ to be this C. We then have (iv) and

$$\begin{aligned} \{\lambda u f x.\,u(\underline{R})\}(\underline{\Theta},\underline{K},\underline{m}_{C_0})\ &\textbf{conv}\ \ \underline{\Theta}(\{\lambda u f x.\,\underline{R}\}(\underline{\Theta},\underline{K},\underline{m}_{C_0})) \\ &\textbf{conv}\ \ \underline{m}_{C[\lambda u f x.\,u(\underline{R})]} \end{aligned}$$

This completes the proof of the properties (i) - (iv). From (ii), (iii), (iv) the facts that C_0 is valid and that C' is valid when C is valid we infer that $C[\underline{A}]$ is valid for each C-K ordinal formula $\underline{A}$; also that there are more formulae provable in $C[\underline{B}]$ than in $C[\underline{A}]$ when $\underline{A} < \underline{B}$. The truth of our assertions regarding $\underline{N}$ follows now in view of (i) and the definitions of $\underline{N}$ and $\underline{G}$.

We cannot conclude that $\underline{N}$ is an ordinal logic, since the formulae $\underline{A}$ were C-K ordinal formulae, but the formula H enables us to obtain an ordinal logic from $\underline{N}$. By the use of the formula Gr we obtain a formula $\overline{Th}$ such that if $\underline{A}$ has a normal form then $Tn(\underline{A})$ enumerates the G.Rs. of the formulae into which $\underline{A}$ is convertible. Also there is a formula Ck such that if h is the G.R. of a formula $H(\underline{B})$ then $Ck(\underline{h})$ conv $\underline{B}$, but otherwise $Ck(\underline{h})$ conv U. Since $H(\underline{B})$ is an ordinal formula only if $\underline{B}$ is a C-K ordinal formula, $Ck(Tn(\underline{\Omega}, \underline{n}))$ is a C-K ordinal formula for each ordinal formula $\underline{\Omega}$ and integer n. For many ordinal formulae it will be convertible to U, but for suitable $\underline{\Omega}$, n it will be convertible to any given C-K ordinal formula. If we put

$$\underline{\Lambda} \rightarrow \lambda \omega a . \Gamma(\lambda n . \underline{N}(Ck(Tn(\omega, n)), a)$$

$\underline{\Lambda}$ will be the required ordinal logic. In fact on account of the properties of Γ, $\underline{\Lambda}(\underline{\Omega}, \underline{A})$ will be convertible to 2 if and only if there is a positive integer n such that

$$\underline{N}(Ck(Tn(\underline{\Omega}, \underline{n})), \underline{A}) \text{ conv } 2$$

If $\underline{\Omega}$ conv $H(\underline{B})$ there will be an integer n such that

$Ck(Tn(\underline{\Omega}, \underline{n}))$ conv $\underline{B}$, and then

$\underline{N}(Ck(Tn(\underline{\Omega}, \underline{n})), \underline{A})$ conv $\underline{N}(\underline{B}, \underline{A})$

For any n, $Ck(Tn(\underline{\Omega}, \underline{n}))$ is convertible to U or to some $\underline{B}$ where $\underline{\Omega}$ conv $H(\underline{B})$. Thus $\underline{\Lambda}(\underline{\Omega}, \underline{A})$ conv 2 if $\underline{\Omega}$ conv $H(\underline{B})$ and $\underline{N}(\underline{B}, \underline{A})$ conv 2 or if $\underline{N}(U, \underline{A})$ conv 2, but not in any other case.

We may now specialize and consider particular classes W of systems. First let us try to construct the ordinal logic described roughly in the introduction. For W we take the class of systems arising from the system of Principia Mathematica[16] by adjoining to

[16] Whitehead and Russell [1]. The axioms and rules of procedure of a similar system P will be found in a convenient form in Gödel [1]. I shall follow Gödel. The symbols for the natural numbers in P are $0, f0, ff0, \ldots f^{(n)}0 \ldots$ Variables with the suffix 'o' stand for natural numbers.

it axiomatic (in the sense described on p.10) sets of axioms[17].

[17] It is sometimes regarded as necessary that the set of axioms used be computable, the intention being that it should be possible to verify of a formula reputed to be an axiom whether it really is so. We can obtain the same effect with axiomatic sets of axioms in this way. In the rules of procedure describing which are the axioms we incorporate a method of enumerating them, and we also introduce a rule that in the main part of the deduction whenever we write down an axiom as such we must also write down its position in the enumeration. It is possible to verify whether this has been done correctly.

Gödel has shown that primitive recursive relations[18] can be expressed

[18] A relation $F(m_1, \ldots, m_r)$ is primitive recursive if it is the necessary and sufficient condition for the vanishing of a primitive recursive function $\varphi(m_1, \ldots, m_r)$.

by means of formulae in P. In fact there is a rule whereby given the recursion equations defining a primitive recursive relation we can find a formula[19] $\mathfrak{A}[x_0, \ldots, z_0]$ such that $\mathfrak{A}[f^{(m_1)}0, \ldots, f^{(m_r)}0]$

[19] Capital German letters will be used to stand for variable or undetermined formulae in P. An expression such as $\mathfrak{A}[\mathfrak{B}, \mathfrak{C}]$ will stand for the result of substituting $\mathfrak{B}$ and $\mathfrak{C}$ for x_0 and y_0 in $\mathfrak{A}$.

is provable in P if $F(m_1, \ldots, m_r)$ is true, and its negation is provable otherwise. Further there is a method by which one can tell of a formula $\mathfrak{A}[x_0, \ldots, z_0]$ whether it arises from a primitive recursive relation in this way, and by which one can find the equations which defined the relation. Formulae of this kind will be called recursion formulae. We shall make use of a property they have, which we cannot prove formally here without giving their definition in full, but which is essentially trivial. $\mathrm{Db}[x_0, y_0]$ is to stand for a certain recursion formula such that $\mathrm{Db}[f^{(m)}0, f^{(n)}0]$ is provable in P if $m = 2n$ and its negation is provable otherwise. Suppose that $\mathfrak{A}[x_0]$, $\mathfrak{B}[x_0]$ are two recursion formulae. Then the theorem I am assuming is that there is a recursion relation $\mathfrak{L}_{\mathfrak{A},\mathfrak{B}}[x_0]$ such that we can prove

$$\mathfrak{L}_{\mathfrak{A},\mathfrak{B}}[x_0] \equiv (\exists y_0)\Big(\big(\mathrm{Db}[x_0, y_0] . \mathfrak{A}[y_0]\big) \vee \big(\mathrm{Db}[fx_0, fy_0] . \mathfrak{B}[y_0]\big)\Big) \tag{8.1}$$

in P.

The significant formulae in any of our extensions of P are those of the form

$$(x_0)(\exists y_0)\, \mathfrak{A}[x_0, y_0] \tag{8.2}$$

where $\mathfrak{A}[x_0, y_0]$ is a recursion formula, arising from the relation $R(m, n)$ let us say. The corresponding number theoretic theorem states that for each natural number m there is a natural number n such that $R(m, n)$ is true.

The systems in W which are not valid are those in which a formula of form (8.2) is provable, but at the same time there is a natural number, m say, such that for each natural number n, $R(m, n)$ is false. This means to say that $\sim \mathfrak{A}[f^{(m)}0, f^{(n)}0]$ is provable for each natural number n. Since (8.2) is provable

$(\exists y_0)\, \mathfrak{A}[f^{(m)}0,\, y_0]$ is provable, so that

$$(\exists y_0)\, \mathfrak{A}[f^{(m)}0, y_0], \sim \mathfrak{A}[f^{(m)}0, 0], \sim \mathfrak{A}[f^{(m)}0, f0] \ldots \tag{8.3}$$

are all provable in the system. We may simplify (8.3). For a given m we may prove a formula of form $\mathfrak{A}[f^{(m)}0, y_0] \equiv \mathfrak{B}[y_0]$ in P, where $\mathfrak{B}[y_0]$ is a recursion formula. Thus we find that the necessary and sufficient condition for a system of W to be valid is that for no recursion formula $\mathfrak{B}[x_0]$ are all of the formulae

$$(\exists x_0)\, \mathfrak{B}[x_0], \sim \mathfrak{B}[0], \sim \mathfrak{B}[f0], \ldots \tag{8.4}$$

provable. An important consequence of this is that if
$\mathfrak{A}_1[x_0],\ \mathfrak{A}_2[x_0], \ldots, \mathfrak{A}_n[x_0]$
are recursion formulae and

$$(\exists x_0)\mathfrak{A}_1[x_0] \vee (\exists x_0)\mathfrak{A}_2[x_0] \vee \ldots \vee (\exists x_0)\, \mathfrak{A}_n[x_0] \tag{8.5}$$

is provable in C, and C is valid, then we can prove $\mathfrak{A}_r[f^{(a)}0]$ in C for some natural numbers r, a where $1 \leqslant r \leqslant n$. Let us define $\mathfrak{D}_r$ to be the formula
$(\exists x_0)\, \mathfrak{A}_1[x_0] \vee \ldots \vee (\exists x_0)\, \mathfrak{A}_r[x_0]$
and define $\mathfrak{E}_r[x_0]$ recursively by the condition that $\mathfrak{E}_1[x_0]$ be $\mathfrak{A}_1[x_0]$ and $\mathfrak{E}_{r+1}[x_0]$ be $\mathfrak{L}_{\mathfrak{E}_r, \mathfrak{A}_{r+1}}[x_0]$. Now I say that

$$\mathfrak{D}_r \supset (\exists x_0)\ \mathfrak{E}_r[x_0] \tag{8.6}$$

is provable for $1 \leqslant r \leqslant n$. It is clearly provable for $r = 1$; suppose it provable for a given r. We can prove

$$(y_0)(\exists x_0)\ \mathrm{D}b[x_0, y_0]$$

and

$$(y_0)(\exists x_0)\, \mathcal{D}b(f x_0, f y_0)$$

from which we obtain

$$\mathcal{E}_r[y_0] \supset (\exists x_0)((\mathcal{D}b[x_0, y_0] \cdot \mathcal{E}_r[y_0]) \vee (\mathcal{D}b[x_0, y_0] \cdot \mathfrak{A}_{r+1}[y_0]))$$

and

$$\mathfrak{A}_{r+1}[y_0] \supset (\exists x_0)((\mathcal{D}b[x_0, y_0] \cdot \mathcal{E}_r[y_0]) \vee (\mathcal{D}b[x_0, y_0] \cdot \mathfrak{A}_{r+1}[y_0]))$$

These together with (8.1) yield

$$(\exists y_0)\, \mathcal{E}_r[y_0] \vee (\exists y_0)\, \mathfrak{A}_{r+1}[y_0] \supset (\exists x_0)\, \mathcal{L}_{\mathcal{E}_r, \mathfrak{A}_{r+1}}[x_0]$$

which suffices to prove (8.6) for $r+1$. Now since (8.5) is provable in C, $(\exists x_0)\, \mathcal{E}_n[x_0]$ must be also, and since C is valid this means that $\mathcal{E}_n[f^{(m)}0]$ must be provable for some natural number m. From (8.1) and the definition of $\mathcal{E}_n[x_0]$ we see that this implies that $\mathfrak{A}_r[f^{(a)}0]$ is provable for some natural number a, and integer r, $1 \leq r \leq n$.

To any system C of W we can assign a primitive recursive relation $P_C(m, n)$ with the intuitive meaning 'm is the G.R. of a proof of the formula whose G.R. is n'. The corresponding recursion formula is $\mathrm{Proof}_C[x_0, y_0]$ (i.e. $\mathrm{Proof}_C[f^{(m)}0, f^{(n)}0]$ is provable when $P_C(m, n)$ is true, and its negation is provable otherwise). We can now explain what is the relation of a system C' to its predecessor C. The set of axioms which we adjoin to P to obtain C' consists of those adjointed in obtaining C, together with all formulae of the form

$$(\exists x_0)\, \mathrm{Proof}_C[x_0, f^{(m)}0] \supset \mathcal{F} \tag{8.7}$$

where m is the G.R. of $\mathcal{F}$.

p. 53 further on.

is provable in C . But by repetition of a previous argument this means that $\mathfrak{A}'_{\ell}$ is provable for some ℓ, $1 \leq \ell \leq k'$ contrary to hypothesis. This is the required contradiction.

We may now construct an ordinal logic in the manner described on p. 44-48 . But let us carry out the construction in rather more detail, and with some modifications appropriate to the particular case. Each system C of our set W may be described by means of a W.F.F. M_C which enumerates the G.Rs. of the axioms of C . There is a W.F.F. E such that if a is the G.R. of some proposition $\mathfrak{F}$ then $E(M_C, \underline{a})$ is convertible to the G.R. of

$$(\exists x_0)\, \mathrm{Proof}_C[x_0, f^{(a)}0] \supset \mathfrak{F}$$

If a is not the G.R. of any proposition in P then $E(M_C, \underline{a})$ is to be convertible to the G.R. of 0 = 0. From E we obtain a W.F.F. K such that $K(M_C, 2\underline{n}+1)$ conv $M_C(\underline{n})$, $K(M_C, 2\underline{n})$ conv $E(M_C, \underline{n})$. The successor system C' is defined by $K(M_C)$ conv $M_{C'}$. Let us choose a formula G such that $G(M_C, \underline{A})$ conv 2 if and only if the number theoretic theorem equivalent to '$\underline{A}$ is dual' is provable in C . Then we define Λ_P by

$$\Lambda_P \rightarrow \lambda \omega a . \Gamma(\lambda y . G(Ck(Tn(\omega, y), \lambda m n . m(\varpi(2,n), \varpi(3,n)), K, M_P)), a)$$

This is an ordinal logic provided that P is valid.

Another ordinal logic of this type has in effect been introduced by Church[20]. Superficially this ordinal logic seems to have no more

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[20] In outline Church [1], 279-280. In greater detail Church [2], Chap. X.

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in common with Λ_P than that they both arise by the method we have described which uses C-K ordinal formulae. The initial systems are entirely different. However, in the relation between C and C' there is an interesting analogy. In Church's method the step from C to C' is performed by means of subsidiary axioms of which the most important (Church [2], p. 88, 1_m) is almost a direct translation into his symbolism of the rule that we may take any formula of form (8.4) as an axiom. There are other extra axioms, however, in Church's system, and it is therefore not unlikely that it is in some sense more complete than Λ_P.

There are other types of ordinal logic, apparently quite unrelated to the type we have so far considered. I have in mind two types of ordinal logic, both of which can be best described directly in terms of ordinal formulae without any reference to C-K ordinal formulae. I shall describe here a specimen of one type, suggested by Hilbert (Hilbert [1], 183ff), and leave the other type over to § 12.

Suppose we have selected a particular ordinal formula $\underline{\Omega}$. We shall construct a modification $P_{\underline{\Omega}}$ of the system P of Gödel (see footnote 16). We shall say that a natural number n is a <u>type</u> if it is either even or $2p-1$ where $\underline{\Omega}(\underline{p}, \underline{p})$ conv 3. The definition of a variable in P is to be modified by the condition that the only admissible subscripts are to be the types in our sense. Elementary expressions are then defined as in P: in particular the definition of an elementary expression of type 0 is unchanged. An elementary formula is defined to be a sequence of symbols of the form $\mathfrak{A}_m \mathfrak{A}_n$ where $\mathfrak{A}_m$, $\mathfrak{A}_n$ are elementary expressions of types m, n satisfying one of the conditions (a), (b), (c).

(a) m and n are both even and m exceeds n,

(b) m is odd and n is even,

(c) $m = 2p - 1$, $n = 2q - 1$ and $\underline{\Omega}(\underline{p}, \underline{q})$ conv 1.

With these modifications the formal development of $\mathbf{P}_{\underline{\Omega}}$ is the same as that of $\mathbf{P}$. We wish however to have a method of associating number theoretic theorems with certain of the formulae of $\mathbf{P}_{\underline{\Omega}}$. We cannot take over directly the association we used in $\mathbf{P}$. Suppose G is a formula in $\mathbf{P}$ interpretable as a number theoretic theorem in the way we described when constructing Λ_P (p. 50). Then if every type suffix in G is doubled we shall obtain a formula in $\mathbf{P}_{\underline{\Omega}}$ which is to be interpreted as the same number theoretic theorem. By the method of § 6 we can now obtain from $\mathbf{P}_{\underline{\Omega}}$ a formula $L_{\underline{\Omega}}$ which is a logic formula of $\mathbf{P}_{\underline{\Omega}}$ is valid; in fact given $\underline{\Omega}$ there is a method of obtaining $L_{\underline{\Omega}}$, so that there is a formula Λ_H such that $\Lambda_H(\underline{\Omega})$ conv $L_{\underline{\Omega}}$ for each ordinal formula $\underline{\Omega}$.

Having now familiarized ourselves with ordinal logics by means of these examples we may begin to consider general questions concerning them.

We wish to show that a contradiction can be obtained by assuming C' to be invalid but C to be valid. Let us suppose that a set of formulae of form (8.4) is provable in C'. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, . . . $\mathfrak{A}_k$ be those axioms of C' of form (8.7) which are used in the proof of $(\exists x_0)\,\mathfrak{B}[x_0]$. We may suppose that none of them are provable in C. Then by the deduction theorem we see that

$$(\mathfrak{A}_1 \cdot \mathfrak{A}_2 \ldots \mathfrak{A}_k) \supset (\exists x_0)\,\mathfrak{B}[x_0] \qquad (8.8)$$

is provable in C. Let $\mathfrak{A}_l$ be $(\exists x_0)\,\mathrm{Proof}_C[x_0, f^{(m_l)}0] \supset \mathfrak{F}_l$. Then from (8.8) we find that

$$(\exists x_0)\,\mathrm{Proof}_C[x_0, f^{(m_1)}0] \vee \ldots \vee (\exists x_0)\,\mathrm{Proof}_C[x_0, f^{(m_k)}0] \vee (\exists x_0)\,\mathfrak{B}[x_0]$$

is provable in C. It follows from a result we have just proved that either $\mathfrak{B}[f^{(c)}0]$ is provable for some natural number c, or else $\mathrm{Proof}_C[f^{(u)}0, f^{(m_l)}0]$ is provable in C for some natural number u and some l, $1 \leqslant l \leqslant k$: but this would mean that $\mathfrak{F}_l$ was provable in C (this is one of the points where we assume the validity of C) and therefore also in C', contrary to hypothesis. Then $\mathfrak{B}[f^{(c)}0]$ must be provable in C; but we are also assuming $\sim \mathfrak{B}[f^{(c)}0]$ is provable in C'. There is therefore a contradiction in C'. Let us suppose that the axioms $\mathfrak{A}_1' \ldots \mathfrak{A}_{k'}'$ of form (8.7) when adjoined to C suffice to obtain the contradiction and that none of these axioms are provable in C. Then

$$\sim \mathfrak{A}_1' \vee \sim \mathfrak{A}_2' \vee \ldots \vee \sim \mathfrak{A}_{k'}'$$

is provable in C, and if $\mathfrak{A}_l'$ is $(\exists x_0)\,\mathrm{Proof}_C[x_0, f^{(m_l')}0] \supset \mathfrak{F}_l'$ then

$$(\exists x_0)\,\mathrm{Proof}_C[x_0, f^{(m_1')}0] \vee \ldots \vee (\exists x_0)\,\mathrm{Proof}[x_0, f^{(m_{k'}')}0]$$

9. Completeness questions.

The purpose of introducing ordinal logics was to avoid as far as possible the effects of Gödel's theorem. It is a consequence of this theorem, suitably modified, that it is impossible to obtain a complete logic formula, or (roughly speaking now) a complete system of logic. We were able, however, from a given system to obtain a more complete one by the adjunction as axioms of formulae, seen intuitively to be correct, but which the Gödel theorem shows are unprovable[21] in the original system; from this we obtained a yet more complete system by a repetition of the process, and so on. We found that the repetition of the process gave us a new system for each C-K ordinal formula. We should like to know whether this process suffices, or whether the system should be extended in other ways as well. If it were possible to tell of a W.F.F. in normal form whether it was an ordinal formula we should know for certain that it was necessary to extend in other ways. In fact for any ordinal formula Λ it would then be possible to find a single logic formula L such that if $\Lambda(\Omega, A)$ conv 2 for some ordinal formula Ω then $L(A)$ conv 2. Since L must be incomplete there must be formulae A for which $\Lambda(\Omega, A)$ is not convertible to 2 for any ordinal formula Ω. However, in view of the fact, proved in § 7, that there is no method of determining of a formula in normal form whether it is an ordinal formula, the case does not arise, and there is still a possibility that some

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21 In the case of P we adjoin all of the axioms $(\exists x_0)\, \mathrm{Proof}_P[x_0, f^{(m)}0]$, where m is the G.R. of f, some of which the Gödel theorem shows to be unprovable in P.

ordinal logics may be complete in some sense. There is quite a natural way of defining completeness.

<u>Definition of completeness of an ordinal logic</u>. We say that an ordinal logic $\underline{\Lambda}$ is complete if for each dual formula $\underline{A}$ there is an ordinal formula $\underline{\Omega}_A$ such that $\underline{\Lambda}(\underline{\Omega}_A, \underline{A})$ conv 2.

As has been explained in § 2, the reference in the definition to the existence of $\underline{\Omega}_A$ for each $\underline{A}$ is to be understood in the same naive way as any reference to existence in mathematics.

There is room for modification in this definition: we might require that there be a formula $\underline{X}$ such that $\underline{X}(\underline{A})$ conv $\underline{\Omega}_A$, $\underline{X}(\underline{A})$ being an ordinal formula whenever $\underline{A}$ is dual. There is no need, however, to discuss the relative merits of these two definitions, because in all cases where we prove an ordinal logic to be complete we shall prove it to be complete even in the modified sense, but in cases where we prove an ordinal logic to be incomplete we use the definition as it stands.

In the terminology of § 6 $\underline{\Lambda}$ is complete if the class of logics $\underline{\Lambda}(\underline{\Omega})$ is complete when $\underline{\Omega}$ runs through all ordinal formulae.

There is another completeness property which is related to this one. Let us for the moment say that an ordinal logic $\underline{\Lambda}$ is <u>all inclusive</u> if to each logic formula $\underline{L}$ there corresponds an ordinal formula $\underline{\Omega}_{(L)}$ such that $\underline{\Lambda}(\underline{\Omega}_{(L)})$ is as complete as $\underline{L}$. Clearly every all inclusive ordinal logic is complete, for if $\underline{A}$ is dual then $S(\underline{A})$ is a logic with $\underline{A}$ in its extent. But if $\underline{\Lambda}$ is complete and

$$A i \rightarrow \lambda k w a . \Gamma(\lambda r . S(4, S(2, k(w, V(Nm(r)))) + S(2, Nm(r, a))))$$

then $Ai(\underline{\Lambda})$ is an all inclusive ordinal logic. For if $\underline{A}$ is in the extent of $\underline{\Lambda}(\underline{\Omega}_{\underline{A}})$ for each $\underline{A}$, and we put $\underline{\Omega}_{(\underline{L})} \rightarrow \underline{\Omega}_{V(\underline{L})}$ then I say that if $\underline{B}$ is in the extent of $\underline{L}$ it must be in the extent of $Ai(\underline{\Lambda}, \underline{\Omega}_{(\underline{L})})$. In fact $Ai(\underline{\Lambda}, \underline{\Omega}_{V(\underline{L})}, \underline{B})$
conv $\Gamma(\lambda r.\, \delta(4, \delta(2, \underline{\Lambda}(\underline{\Omega}_{V(\underline{L})}, V(Nm(r)))) + \delta(2, Nm(r, \underline{B}))))$
For suitable n, $Nm(\underline{n})$ conv $\underline{L}$ and then

$$\underline{\Lambda}(\underline{\Omega}_{V(\underline{L})}, V(Nm(\underline{n}))) \text{ conv } 2$$

$$Nm(\underline{n}, \underline{B}) \text{ conv } 2$$

and therefore by the properties of Γ, δ

$$Ai(\underline{\Lambda}, \underline{\Omega}_{V(\underline{L})}, \underline{B}) \text{ conv } 2$$

Conversely $Ai(\underline{\Lambda}, \underline{\Omega}_{V(\underline{L})}, \underline{B})$ can only be convertible to 2 if both $Nm(\underline{n}, \underline{B})$ and $\underline{\Lambda}(\underline{\Omega}_{V(\underline{L})}, V(Nm(\underline{n})))$ are convertible to 2 for some positive integer n; but if $\underline{\Lambda}(\underline{\Omega}_{V(\underline{L})}, V(Nm(\underline{n})))$ conv 2 then $Nm(\underline{n})$ must be a logic and since $Nm(\underline{n}, \underline{B})$ conv 2, $\underline{B}$ must be dual.

It should be noticed that our definitions of completeness refer only to number theoretic theorems. Although it would be possible to introduce formulae analogous to ordinal logics which would prove more general theorems than number theoretic ones, and have a corresponding definition of completeness, yet if our theorems are too general we shall find that our (modified) ordinal logics are never complete. This follows from the argument of § 4. If our 'oracle' tells us, not whether any given number theoretic statement is true, but whether a given formula is an ordinal formula, the argument still applies, and we find there are classes of problem which cannot be solved by a uniform process even with the help of this oracle. This is equivalent to saying that there is no ordinal logic of the proposed modified type which is complete with respect to these problems. This situation becomes more definite if we take formulae satisfying conditions (a) - (e), (f') (as described at the end of §12) instead of ordinal formulae; it is then not possible for the ordinal logic to be complete with respect to any class of problems more extensive than the number theoretic problems.

We might hope to obtain some intellectually satisfying system of logical inference (for the proof of number theoretic theorems) with some ordinal logic. Gödel's theorem shows that such a system cannot be wholly mechanical, but with a complete ordinal logic we should be able to confine the non-mechanical steps entirely to verifications that particular formulae are ordinal formulae.

We might also expect to obtain an interesting classification of number theoretic theorems according to 'depth'. A theorem which required an ordinal α to prove it would be deeper than one which could be proved by the use of an ordinal β less than α. However, this presupposes more than is justified. We define

<u>Definition of invariance of ordinal logics</u>. An ordinal logic $\underline{\Lambda}$ is said to be <u>invariant up to</u> an ordinal α if, whenever $\underline{\Omega}$, $\underline{\Omega}'$ are ordinal formulae representing the same ordinal less than α, the extent of $\underline{\Lambda}(\underline{\Omega})$ is identical with the extent of $\underline{\Lambda}(\underline{\Omega}')$. An ordinal logic is <u>invariant</u> if it is invariant up to each ordinal represented by an ordinal formula.

Clearly the classification into depths presupposes that the ordinal logic used is invariant.

Among the questions we should now like to ask are

(a) are there any complete ordinal logics?

(b) are there any complete invariant ordinal logics?

To these we might have added 'are all ordinal logics complete?'; but this is trivial; in fact there are ordinal logics which do not suffice to prove any number theoretic theorems whatever.

We shall now show that (a) must be answered affirmatively. In fact we can write down a complete ordinal logic at once. Put

$$Od \rightarrow \lambda a . \{\lambda f m n . \mathrm{Dt}(f(m), f(n))\}(\lambda s . \mathcal{P}(\lambda r . r(I, a(s)), 1, s))$$

and

$$Comp \rightarrow \lambda w a . \delta(w, Od(a))$$

I shall show that $Comp$ is a complete ordinal logic.

~~In fact~~ For if $Comp(\underline{\Omega}, \underline{A})$ conv 2, then

$\underline{\Omega}$ conv $Od(\underline{A})$

conv $\lambda m n . \mathrm{Dt}(\mathcal{P}(\lambda r . r(I, \underline{A}(m)), 1, m)), \mathcal{P}(\lambda r . r(I, \underline{A}(n)), 1, n))$

$\underline{\Omega}(\underline{m}, \underline{n})$ has a normal form if $\underline{\Omega}$ is an ordinal formula, so that then $\mathcal{P}(\lambda r . r(I, \underline{A}(\underline{m})), 1)$ has a normal form; this means that $\underline{r}(I, \underline{A}(\underline{m}))$ conv 2 some r, i.e. $\underline{A}(\underline{m})$ conv 2. Thus if $Comp(\underline{\Omega}, \underline{A})$ conv 2 and $\underline{\Omega}$ is an ordinal formula then $\underline{A}$ is dual. $Comp$ is therefore an ordinal logic. Now suppose conversely that $\underline{A}$ is dual. I shall show that $Od(\underline{A})$ is an ordinal formula representing the ordinal ω. ~~In fact~~ For

$$\mathcal{P}(\lambda r . r(I, \underline{A}(\underline{m})), 1, \underline{m}) \text{ conv } \mathcal{P}(\lambda r . r(I, 2), 1, \underline{m})$$

$$\text{conv } I(\underline{m}) \text{ conv } \underline{m}$$

$$Od(\underline{A}, \underline{m}, \underline{n}) \text{ conv } Dt(\underline{m}, \underline{n})$$

i.e. $Od(\underline{A})$ is an ordinal formula representing the same ordinal as Dt. But

$$Comp(Od(\underline{A}), \underline{A}) \text{ conv } S(Od(\underline{A}), Od(\underline{A})) \qquad \text{conv } 2$$

This proves the completeness of $Comp$.

Of course $Comp$ is not the kind of complete ordinal logic that we should really want to use. The use of $Comp$ does not make it any easier to see that $\underline{A}$ is dual. ~~In fact~~ if we really want to use an ordinal logic a proof of completeness for that particular ordinal logic will be of little value; the ordinals given by the completeness proof will not be ones which can ~~really~~ easily be seen intuitively to be ordinals. The only value ~~of~~ a completeness proof of this kind would have would be to show that if any objection is to be raised against an ordinal logic it must be on account of something more subtle then incompleteness.

The theorem of completeness is also unexpected in that the ordinal formulae used are all formulae representing ω. This is contrary to our intentions in constructing Λ_P for instance; implicitly we had in mind large ordinals expressed in a simple manner Here we have small ordinals expressed in a very complex and artificial way.

Before trying to solve the problem (b), let us see how far Λ_P and Λ_T are invariant. We should certainly not expect Λ_P to be invariant, as the extent of $\Lambda_P(\underline{\Omega})$ wil depend on whether $\underline{\Omega}$

is convertible to a formula of form $H(\underline{A})$: but suppose we call an ordinal logic $\underline{\Lambda}$ C-K invariant up to α if the extent of $\underline{\Lambda}(H(\underline{A}))$ is the same as the extent of $\underline{\Lambda}(H(\underline{B}))$ whenever $\underline{A}$ and $\underline{B}$ are C-K ordinal formulae representing the same ordinal less than α. How far is $\underline{\Lambda}_P$ C-K invariant? It is not difficult to see that it is C-K invariant up to any finite ordinal, that is to say up to ω. It is also C-K invariant up to $\omega+1$, and follows from the fact that the extent of $\underline{\Lambda}_P(H(\lambda u f x . u(\underline{R})))$ is the set theoretic sum of the extents of

$$\underline{\Lambda}_P(H(\lambda u f x . \underline{R}(1))),\ \underline{\Lambda}_P(H(\lambda u f x . \underline{R}(2))),\ \ldots$$

However, there is no obvious reason to believe that it is C-K invariant up to $\omega+2$, and in fact it is demonstrable that this is not the case (see the end of this section). Let us try to see what happens if we try to prove that the extent of $\underline{\Lambda}_P(H(Suc(\lambda u f x . u(\underline{R}_1))))$ is the same as the extent of $\underline{\Lambda}_P(H(Suc(\lambda u f x . u(\underline{R}_2))))$ where $\lambda u f x . u(\underline{R}_1)$ and $\lambda u f x . u(\underline{R}_2)$ are two C-K ordinal formulae representing ω. We should have to prove that a formula interpretable as a theorem of number theory is provable in $C[Suc(\lambda u f x . u(\underline{R}_1))]$ if and only if it is provable in $C[Suc(\lambda u f x . u(\underline{R}_2))]$. Now $C[Suc(\lambda u f x . u(\underline{R}_1))]$ is obtained from $C[\lambda u f x . u(\underline{R}_1)]$ by adjoining all axioms of form

$$(\exists x_0)\, Proof_{C[\lambda u f x . u(\underline{R}_1)]}[x_0, f^{(m)}0] \supset \mathfrak{F} \tag{9.1}$$

where m is the G.R. of $\mathfrak{F}$, and $C[Suc(\lambda u f x . u(\underline{R}_2))]$ is obtained from $C[\lambda u f x . u(\underline{R}_2)]$ by adjoining all axioms of form

$$(\exists x_0)\, Proof_{C[\lambda u f x . u(\underline{R}_2)]}[x_0, f^{(m)}0] \supset \mathfrak{F} \tag{9.2}$$

The axioms which must be adjoined to P to obtain $C[\lambda u f x . u(\underline{R}_1)]$ are essentially the same as those which must be adjoined to obtain $C[\lambda u f x . u(R_2)]$: however the <u>rules of procedure which have to be applied before these axioms can be written down will in general be quite different in the two cases</u>. Consequently (9.1) and (9.2) will be quite different axioms, and there is no reason to expect their consequences to be the same. A proper understanding of this will make our treatment of question (b) much more intelligible. See also footnote .

Now let us turn to Λ_T. This ordinal logic is invariant. Suppose $\underline{\Omega}$, $\underline{\Omega}'$ represent the same ordinal, and suppose we have a proof of a number theoretic theorem G in $P_{\underline{\Omega}}$. The formula expressing the number theoretic theorem does not involve any odd types. Now there is a one-one correspondence between the odd types such that if $2m-1$ corresponds to $2m'-1$ and $2n-1$ to $2n'-1$ then $\underline{\Omega}(\underline{m},\underline{n})$ conv 2 implies $\underline{\Omega}'(\underline{m}',\underline{n}')$ conv 2. Let us modify the odd type-subscripts occurring in the proof of G, replacing each by its mate in the one-one correspondence. There results a proof in $P_{\underline{\Omega}'}$ with the same end formula G. That is to say that if G is provable in $P_{\underline{\Omega}}$ it is provable in $P_{\underline{\Omega}'}$; Λ_T is invariant.

The question (b) must be answered negatively. Much more can be proved, but we shall first prove an even weaker result which can be established very quickly, in order to illustrate the method.

I shall prove that an ordinal logic $\underline{\Lambda}$ cannot be invariant and have the property that the extent of $\underline{\Lambda}(\underline{\Omega})$ is a strictly increasing function of the ordinal represented by $\underline{\Omega}$. Suppose $\underline{\Lambda}$ has these properties; we shall obtain a contradiction. Let $\underline{A}$ be a W.F.F. in normal form and without free variables, and consider the process of carrying out conversions on $\underline{A}(1)$ until we have shown it convertible to 2, then converting $\underline{A}(2)$ to 2, then $\underline{A}(3)$ and so on: suppose that after r steps we are still performing the conversion on $\underline{A}(\underline{m}_r)$. There is a formula Jh such that $\mathrm{Jh}(\underline{A}, \underline{r})$ conv $\underline{m}_r$ for each positive integer r . Now let Z be a formula such that for each positive integer n , $Z(\underline{n})$ is an ordinal formula representing ω^n , and suppose $\underline{B}$ is a member of the extent of $\underline{\Lambda}(\mathrm{Suc}(\mathrm{Lim}(Z)))$ but not of the extent of $\underline{\Lambda}(\mathrm{Lim}(Z))$. Put

$$\underline{K}^* \to \lambda a.\, \underline{\Lambda}(\mathrm{Suc}(\mathrm{Lim}(\lambda r.\, Z(\mathrm{Jh}(a, r)))), \underline{B})$$

then $\underline{K}^*$ is a complete logic. For if $\underline{A}$ is dual, then $\mathrm{Suc}(\mathrm{Lim}(\lambda r.\, Z(\mathrm{Jh}(\underline{A}, r))))$ represents the ordinal $\omega^\omega + 1$, and therefore $\underline{K}^*(\underline{A})$ conv 2; but if $\underline{A}(\underline{u})$ is not convertible to 2, then $\mathrm{Suc}(\mathrm{Lim}(\lambda r.\, Z(\mathrm{Jh}(\underline{A}, r))))$ represents an ordinal not exceeding $\omega^u + 1$, and $\underline{K}^*(\underline{A})$ is therefore not convertible to 2. Since there are no complete logic formulae this proves our assertion.

We may now prove more powerful results.

<u>Incompleteness theorems</u>. (A) If an ordinal logic $\underline{\Lambda}$ is invariant up to an ordinal α , then for any ordinal formula $\underline{\Omega}$ representing an ordinal β , $\beta < \alpha$, the extent of $\underline{\Lambda}(\underline{\Omega})$ is contained in the (set-theoretic) sum of the extents of the logics $\underline{\Lambda}(P)$ where P is finite.

(B) If an ordinal logic $\underline{\Lambda}$ is C-K invariant up to an ordinal α, then for any C-K ordinal formula $\underline{A}$ representing an ordinal β, $\beta<\alpha$, the extent of $\underline{\Lambda}(H(\underline{A}))$ is contained in the (set-theoretic) sum of the extents of the logics $\Lambda(H(\underline{F}))$ where $\underline{F}$ is a C-K ordinal formula representing an ordinal less than ω^2.

<u>Proof of (A)</u>. It suffices to prove that if $\underline{\Omega}$ represents an ordinal γ, $\omega \leq \gamma < \alpha$, then the extent of $\underline{\Lambda}(\underline{\Omega})$ is contained in the set theoretic sum of the extents of the logics $\underline{\Lambda}(\underline{\Omega}')$ where $\underline{\Omega}'$ represents an ordinal less than γ. The ordinal γ must be of the form $\gamma_0+\rho$ where ρ is finite and represented by $\underline{P}$ say, and γ_0 is not the successor of any ordinal and is not less than ω. There are two cases to consider; $\gamma_0=\omega$ and $\gamma_0 \geqslant 2\omega$. In each of them we shall obtain a contradiction from the assumption that there is a W.F.F. $\underline{B}$ such that $\underline{\Lambda}(\underline{\Omega},\underline{B})$ conv 2 whenever $\underline{\Omega}$ represents γ, but is not convertible to 2 if $\underline{\Omega}$ represents a smaller ordinal. Let us take first the case $\gamma_0 \geqslant 2\omega$. Suppose $\gamma_0=\omega+\gamma_1$, and that $\underline{\Omega}_1$ is an ordinal formula representing γ_1. Let $\underline{A}$ be any W.F.F. with a normal form and no free variables, and let Z be the class of those positive integers which are exceeded by all integers n for which $\underline{A}(\underline{n})$ is not convertible to 2. Let E be the class of integers $2p$ such that $\underline{\Omega}(\underline{p},\underline{n})$ conv 2 for some n belonging to Z. The class E, together with the class φ of all odd integers is constructively enumerable. It is evident that the class can be enumerated with repetitions, and since it is infinite the required enumeration can be obtained by striking out the repetitions. There is, therefore, a formula $\underline{E}w$ such that $\underline{E}w(\underline{\Omega},\underline{A},\underline{r})$ runs through the formulae of the class $E+\varphi$ without repetitions as r runs through the positive integers. We define

$$\underline{Rt} \rightarrow \lambda \varpi a m n .\ Sum(\underline{\mathfrak{D}t}, \varpi, \underline{En}(\varpi, a, m), \underline{En}(\varpi, a, n))$$

Then $\underline{Rt}(\underline{\Omega}_1, \underline{A})$ is an ordinal formula which represents γ_0 if $\underline{A}$ is dual, but a smaller ordinal otherwise. In fact

$\underline{Rt}(\underline{\Omega}_1, \underline{A}, \underline{m}, \underline{n})$ conv $\{Sum(\underline{\mathfrak{D}t}, \underline{\Omega}_1)\}(\underline{En}(\underline{\Omega}_1, \underline{A}, \underline{m}), \underline{En}(\underline{\Omega}_1, \underline{A}, \underline{n}))$
Now if $\underline{A}$ is dual $E + \varphi$ includes all integers m for which $\{Sum(\underline{\mathfrak{D}t}, \underline{\Omega}_1)\}(\underline{m}, \underline{m})$ conv 3. Putting "$\underline{En}(\underline{\Omega}_1, \underline{A}, \underline{p})$ conv q" for $M(p, q)$ we see that condition (7.4) is satisfied, so that $\underline{Rt}(\underline{\Omega}_1, \underline{A})$ is an ordinal formula representing γ_0. But if $\underline{A}$ is not dual the set $E + \varphi$ consists of all integers m for which $\{Sum(\underline{\mathfrak{D}t}, \underline{\Omega}_1)\}(\underline{m}, \underline{r})$ conv 2, where r depends only on $\underline{A}$. In this case $\underline{Rt}(\underline{\Omega}_1, \underline{A})$ is an ordinal formula representing the same ordinal as $\underline{Inf}(Sum(\underline{\mathfrak{D}t}, \underline{\Omega}_1), \underline{r})$, and this is smaller than γ_0. Now consider $\underline{K}$:

$$\underline{K} \rightarrow \lambda a.\ \underline{\Lambda}(Sum(\underline{Rt}(\underline{\Omega}_1, \underline{A}), \underline{P}), \underline{B})$$

If $\underline{A}$ is dual, $\underline{K}(\underline{A})$ is convertible to 2, since $Sum(\underline{Rt}(\underline{\Omega}_1, \underline{A}), \underline{P})$ represents γ. But if $\underline{A}$ is not dual it is not convertible to 2, for $Sum(\underline{Rt}(\underline{\Omega}_1, \underline{A}), \underline{P})$ then represents an ordinal smaller than γ. In $\underline{K}$ we therefore have a complete logic formula, which is impossible.

Now we take the case $\gamma_0 = \omega$. We introduce a W.F.F. $\underline{Mg}$ such that if n is the D.N. of a computing machine $\mathcal{M}$, and if by the mth complete configuration of $\mathcal{M}$ the figure 0 has been printed then $\underline{Mg}(\underline{n}, \underline{m})$ is convertible to $\lambda pq.\ \underline{Al}(4(\underline{P}, 2p + 2q), 3, 4)$ (which is an ordinal formula representing the ordinal 1), but if 0 has not been printed it is convertible to $\lambda pq.\ p(q, \underline{I}, 4)$

(which represents 0). Now consider $\underline{M}$.

$$\underline{M} \to \lambda n.\ \underline{\Lambda}\left(Sum\left(Lim\left(Mg(n)\right), \underline{P}\right), \underline{B}\right)$$

If the machine never prints 0 then $Lim\left(\lambda r.\ Mg(\underline{n}, r)\right)$ represents ω and $Sum\left(Lim\left(Mg(\underline{n})\right), \underline{P}\right)$ represents γ. This means that $Mg(\underline{n})$ is convertible to 2. If, however $\mathcal{M}$ ever prints 0, $Sum\left(Lim\left(Mg(\underline{n})\right), \underline{P}\right)$ represents a finite ordinal and $\underline{M}(\underline{n})$ is not convertible to 2. In $\underline{M}$ we therefore have a means of determining of a machine whether it ever prints 0, which is impossible[22]. (Turing [1], § 8). This completes the proof of (A).

[22] This part of the argument can equally well be based on the impossibility of determining of two W.F.F. whether they are interconvertible. (Church [3], 363.)

Proof of (B). It suffices to prove that if $\underline{C}$ represents an ordinal γ, $\omega^{\omega} \leq \gamma < \alpha$ then the extent of $\underline{\Lambda}(H(\underline{C}))$ is included in the set-theoretic sum of the extents of $\underline{\Lambda}(H(\underline{G}))$ where $\underline{G}$ represents an ordinal less than γ. We obtain a contradiction from the assumption that there is a formula $\underline{B}$ which is in the extent of $\underline{\Lambda}(H(\underline{G}))$ if $\underline{G}$ represents γ, but not if it represents any smaller ordinal. The ordinal γ is of the form $\delta + \omega^{\omega} + \xi$ where $\xi < \omega^{\omega}$. Let $\underline{D}$ be a C-K ordinal formula representing δ and $\underline{\varphi}$ one representing ξ.

We now define a formula H_{g}. Suppose $\underline{A}$ is a W.F.F. in normal form and without free variables; consider the process of carrying out conversions on $\underline{A}(\underline{1})$ until it is brought into the form 2, then converting $\underline{A}(2)$ to 2, then $\underline{A}(3)$, and so on. Suppose that at the r th step of this process we are doing the n_rth step in the conversion of $\underline{A}(\underline{m}_r)$. Thus for instance if $\underline{A}(3)$ be not convertible to 2, m_r can never exceed 3. Then $Hg(\underline{A},\underline{r})$ is to be convertible to $\lambda f\,.\,f(\underline{m}_r,\underline{n}_r)$ for each positive integer r. Put

$$Sq \rightarrow \lambda dmn\,.\,n(Suc, m(\lambda aufx\,.\,u(\lambda y\,.\,y(Suc, a, u, f, x)), d(u, f, x)))$$
$$\underline{M} \rightarrow \lambda aufx\,.\,\underline{\Phi}(u, f, u(\lambda y\,.\,Hg(a, y, Sq(\underline{D}))))$$
$$\underline{K}_1 \rightarrow \lambda a\,.\,\underline{\Lambda}(\underline{M}(a), \underline{B})$$

then I say that $\underline{K}_1$ is a complete logic formula. $Sq(\underline{D}, \underline{m}, \underline{n})$ is a C-K ordinal formula representing $\delta + m\omega + n$, and therefore $Hg(\underline{A}, \underline{r}, Sq(\underline{D}))$ represents an ordinal ζ_r which increases steadily with increasing r, and tends to the limit $\delta + \omega^2$ if $\underline{A}$ is dual. Further $Hg(\underline{A}, \underline{r}, Sq(\underline{D})) < Hg(\underline{A}, S(\underline{r}), Sq(\underline{D}))$ for each positive integer r. $\lambda ufx\,.\,u(\lambda y\,.\,Hg(\underline{A}, y, Sq(\underline{D}))$ is therefore a C-K ordinal formula and represents the limit of the sequence $\zeta_1, \zeta_2, \zeta_3 \ldots$ This is $\delta + \omega^2$ if $\underline{A}$ is dual, but a smaller ordinal otherwise. Likewise $\underline{M}(\underline{A})$ represents γ if $\underline{A}$ is dual, but a smaller ordinal otherwise. The formula $\underline{B}$ therefore belongs to the extent of $\underline{\Lambda}(H(\underline{M}(\underline{A})))$ if and only if $\underline{A}$ is dual, and this implies that $\underline{K}_1$ is a complete logic formula as was asserted. But this is impossible and we have the required contradiction.

As a corollary to (A) we see that Λ_H is incomplete and in fact that the extent of $\Lambda_H(\mathfrak{D}t)$ contains the extent of $\Lambda_H(\underline{\Omega})$ for any ordinal formula $\underline{\Omega}$. This result, suggested to me first by the solution of question (b), may also be obtained more directly. In fact if a number theoretic theorem can be proved in any particular $P_{\underline{\Omega}}$ it can be proved in $P_{\lambda mn.\, m(n,\bar{I},4)}$. The formulae describing number theoretic theorems in P do no involve more than a finite number of types, type 5 being the highest necessary. The formulae describing the number theoretic theorems in any $P_{\underline{\Omega}}$ will be obtained by doubling the type subscripts. Now suppose we have a proof of a number theoretic theorem $\mathfrak{G}$ in $P_{\underline{\Omega}}$ and that the types occurring in the proof are among 0, 2, 4, 6, 8, 10, t_1, t_2, t_3, . . . t_R. We may suppose they have been arranged with all the even types preceding all the odd types, the even types in order of magnitude and the type $2m-1$ preceding $2n-1$ if $\underline{\Omega}(\underline{m},\underline{n})$ conv 2. Now let each t_r be replaced by $10+2r$ throughout the proof of $\mathfrak{G}$. We obtain a proof of $\mathfrak{G}$ in $P_{\lambda mn.\, m(n,\bar{I},4)}$.

As with problem (a) the solution of problem (b) does not require the use of high ordinals (e.g. if we make the assumption that the extent of $\underline{\Lambda}(\underline{\Omega})$ is a steadily increasing function of the ordinal represented by $\underline{\Omega}$ we do not have to consider ordinals higher than $\omega+2$). However, if we restrict what we are to call ordinal formulae in some way we shall have corresponding modified problems (a) and (b); the solutions will presumably be essentially the same but will involve higher ordinals. Suppose for example that Prod is a W.F.F. with the property that $\mathrm{Prod}(\underline{\Omega}_1, \underline{\Omega}_2)$ is an ordinal formula representing $\alpha_1\alpha_2$ when $\underline{\Omega}_1$, $\underline{\Omega}_2$ are ordinal formulae representing α_1, α_2 respectively and suppose we call a W.F.F. a

1-ordinal formula when it is convertible to the form $\mathrm{Sum}(\mathrm{Prod}(\underline{Q}, \mathrm{Dt}), \underline{P})$ where $\underline{Q}$, $\underline{P}$ are ordinal formulae of which $\underline{P}$ represents a finite ordinal. We may define 1-ordinal logics, 1-completeness and 1-invariance in an obvious way, and obtain a solution of problem (b) which differs from the solution in the ordinary case in that the ordinals less than ω take the place of the finite ordinals. More generally the cases I have in mind will be covered by the following theorem.

Suppose we have a class V of formulae representing ordinals in some manner we do not propose to specify definitely, and a subset[23] U of the class V such that

[23] The subset U wholly supersedes V in what follows. The introduction of V serves to emphasise the fact that the set of ordinals represented by members of U may have gaps.

(i) There is a formula $\underline{\Phi}$ such that if $\underline{T}$ enumerates a sequence of members of U representing an increasing sequence of ordinals, then $\underline{\Phi}(\underline{T})$ is a member of U representing the limit of the sequence.

(ii) There is a formula $\underline{E}$ such that $\underline{E}(\underline{m}, \underline{n})$ is a member of U for each pair of positive integers m, n and if it represents $\varepsilon_{m,n}$ then $\varepsilon_{m,n} < \varepsilon_{m',n'}$ if either $m < m'$ or $m = m'$, $n < n'$

(iii) There is a formula $\underline{G}$ such that if $\underline{A}$ is a member of U then $\underline{G}(\underline{A})$ is a member of U representing a larger ordinal than does $\underline{A}$, and such that $\underline{G}(\underline{E}(\underline{m}, \underline{n}))$ always represents an ordinal not larger than $\varepsilon_{m,n+1}$.

We define a V-ordinal logic to be a W.F.F. $\underline{\Lambda}$ such that $\underline{\Lambda}(\underline{A})$ is a logic whenever $\underline{A}$ belongs to V. $\underline{\Lambda}$ is V-invariant if the extent of $\underline{\Lambda}(\underline{A})$ depends only on the ordinal represented by $\underline{A}$. Then it is not possible for a V-ordinal logic $\underline{\Lambda}$ to be V-invariant and have the property that if $\underline{C}_1$ represents a greater ordinal than $\underline{C}_2$, ($\underline{C}_1$ and $\underline{C}_2$ both being members of U) then the extent of $\underline{\Lambda}(\underline{C}_1)$ is greater than the extent of $\underline{\Lambda}(\underline{C}_2)$.

We suppose the contrary. Let $\underline{B}$ be a formula belonging to the extent of $\underline{\Lambda}(\underline{G}(\underline{\Phi}(\lambda r.\, \underline{E}(r, 1))))$ but not to the extent of $\underline{\Lambda}(\underline{\Phi}(\lambda r.\, \underline{E}(r, 1)))$. Suppose that our assertion is false and that

$$\underline{K}' \rightarrow \lambda a.\, \underline{\Lambda}(\underline{\Theta}(\lambda r.\, Hg(a, r, \underline{E})), \underline{B})$$

Then $\underline{K}'$ is a complete logic. For

$$Hg(\underline{A}, \underline{r}, \underline{E}) \quad \text{conv} \quad \underline{E}(\underline{m}_r, \underline{n}_r)$$

$\underline{E}(\underline{m}_r, \underline{n}_r)$ is a sequence of V-ordinal formulae representing an increasing sequence of ordinals. Their limit is represented by $\underline{\Theta}(\lambda r.\, Hg(\underline{A}, r, \underline{E}))$; let us see what this limit is. First suppose $\underline{A}$ is dual: then m_r tends to infinity as r tends to infinity, and $\underline{\Theta}(\lambda r.\, Hg(\underline{A}, r, \underline{E}))$ therefore represents the same ordinal as $\underline{\Theta}(\lambda r.\, \underline{E}(r, 1))$. In this case we must have $\underline{K}'(\underline{A})$ conv 2. Now suppose $\underline{A}$ is not dual: m_r is eventually equal to some constant number, a say, and $\underline{\Theta}(\lambda r.\, Hg(\underline{A}, r, \underline{E}))$ represents the same ordinal as $\underline{\Theta}(\lambda r.\, \underline{E}(\underline{a}, r))$ which is smaller than that represented by $\underline{\Theta}(\lambda r.\, \underline{E}(r, 1))$. $\underline{B}$ cannot therefore belong to the extent of $\underline{\Theta}(\lambda r.\, Hg(\underline{A}, r, \underline{E}))$, and $\underline{K}(\underline{A})$ is not convertible to 2. We have proved that $\underline{K}'$ is a complete logic which is impossible.

This theorem can no doubt be improved in many ways. However, it is sufficiently general to show that, with almost any reasonable notation for ordinals, completeness is incompatible with invariance.

We can still give a certain meaning to the classification into depths with highly restricted kinds of ordinals. Suppose we take a particular ordinal logic $\underline{\Lambda}$ and a particular ordinal formula $\underline{\Psi}$ representing the ordinal α say (preferably a large one), and we restrict ourselves to ordinal formulae of the form $Inf(\underline{\Psi}, \underline{a})$. We shall then have a classification into depths, but the extents of all the logics we so obtain will be contained in the extent of a single logic.

We now attempt a problem of a rather different character, that of the completeness of Λ_P. It is to be expected that this ordinal logic is complete. I cannot at present give a proof of this, but I can give a proof that it is complete as regards a simpler type of theorem than the number theoretic theorems viz. those of form '$\theta(x)$ vanishes identically' where $\theta(x)$ is primitive recursive. The proof will have to be much abbreviated as we do not wish to go into the formal details of the system P. Also there is a certain lack of definiteness in the problem as at present stated, owing to the fact that the formulae G, E, M_P were not completely defined. Our attitude here is that it is open to the sceptical reader to give detailed definitions for these formulae and then verify that the remaining details of the proof can be filled in using his definition. It is not asserted that these details can be filled in whatever be the definitions of G, E, M_P consistent with the properties already required of them, only that it is so with the more natural definitions.

I shall prove the completeness theorem in the following form. If $\mathfrak{G}[x_0]$ is a recursion formula and $\mathfrak{G}[0]$, $\mathfrak{G}[f0]$, . . . are all provable in P, then there is a C-K ordinal formula $\underline{A}$ such that $(x_0)\mathfrak{G}[x_0]$ is provable in the system $P_{\underline{A}}$ of logic obtained from P by adjoining as axioms all formulae whose G.R's are of the form

$$\underline{A}\left(\lambda mn.\, m(\varpi(2,n), \varpi(3,n)), K, M_P, \underline{r}\right)$$

(provided they represent propositions)

First let us define the formula $\underline{A}$. Suppose $\underline{D}$ is a W.F.F. with the property that $\underline{D}(n)$ conv 2 if $\mathfrak{G}[f^{(n-1)}0]$ is provable in P, but $\underline{D}(n)$ conv 1 if $\sim\mathfrak{G}[f^{(n-1)}0]$ is provable in P (P is being assumed consistent). Let Θ be defined by

$$\Theta \rightarrow \{\lambda vu.\, v(v(v,u))\}(\lambda vu.\, v(v(v,u)))$$

and let V be a formula with the properties

$$V(2) \text{ conv } \lambda u.\, u(Suc, U)$$

$$V(1) \text{ conv } \lambda u.\, u(\underline{I}, \Theta(Suc))$$

The existence of such a formula is established in Kleene 1, corollary on p 220. Now put

$$\underline{A}^* \rightarrow \lambda ufx.\, u(\lambda y.\, V(\underline{D}(y), y, u, f, x))$$

$$A \rightarrow Suc(\underline{A}^*)$$

I assert that $\underline{A}^*$, $\underline{A}$ are C-K ordinal fomulae whenever it is true that $\mathfrak{B}[0]$, $\mathfrak{B}[f0]$, . . . are all provable in P. For in this case $\underline{A}^*$ is $\lambda u f x . u(\underline{R})$ where

$$\underline{R} \rightarrow \lambda y . \ V(\underline{D}(y), y, u, f, x)$$

and then

$$\begin{aligned} \lambda u f x . \underline{R}(\underline{n}) \quad &\text{conv} \quad \lambda u f x . \ V(\underline{D}(\underline{n}), \underline{n}, u, f, x) \\ &\text{conv} \quad \lambda u f x . \ V(2, \underline{n}, u, f, x) \\ &\text{conv} \quad \lambda u f x . \ \{\lambda n . n(Suc, U)\}(\underline{n}, u, f, x) \\ &\text{conv} \quad \lambda u f x . \ \underline{n}(Suc, U, u, f, x) \end{aligned}$$

which is a C-K ordinal formula, and

$$\lambda u f x . S(\underline{n}, Suc, U, u, f, x) \ \text{conv} \ Suc(\lambda u f x . \underline{n}(Suc, U, u, f, x))$$

These relations hold for an arbitrary positive integer n and therefore $\underline{A}^*$ is a C-K ordinal formula (condition (9) p. 32): it follows immediately that $\underline{A}$ is also a C-K ordinal formula. It remains to prove that $(x_0)\mathfrak{B}[x_0]$ is provable in $P_{\underline{A}}$. To do this it is necessary to examine the structure of $\underline{A}^*$ in the case that $(x_0)\mathfrak{B}[x_0]$ is false. Let us suppose that $\sim\mathfrak{B}[f^{(a-1)}0]$ is true so that $\underline{D}(\underline{a})$ conv 1, and let us consider $\underline{B}$ where

$$\underline{B} \rightarrow \lambda u f x . \ V(\underline{D}(\underline{a}), \underline{a}, u, f, x)$$

If $\underline{A}^*$ were a C-K ordinal formula then $\underline{B}$ would be a member of its fundamental sequence; but

$$\begin{aligned}
\underline{B} \quad \text{conv} \quad & \lambda u f x . \ V(\underline{1}, \underline{a}, u, f, x) \\
\text{conv} \quad & \lambda u f x . \{\lambda u . u(\underline{I}, \Theta(Suc))\}(\underline{a}, u, f, x) \\
\text{conv} \quad & \lambda u f x . \ \Theta(Suc, u, f, x) \\
\text{conv} \quad & \lambda u f x . \{\lambda u . u(\Theta(u))\}(Suc, u, f, x) \\
\text{conv} \quad & \lambda u f x . \ Suc(\Theta(Suc), u, f, x) \\
\text{conv} \quad & Suc(\lambda u f x . \ \Theta(Suc, u, f, x)) \\
\text{conv} \quad & Suc(\underline{B}) \qquad (9.3)
\end{aligned}$$

This of course implies that $\underline{B} < \underline{B}$ and therefore that $\underline{B}$ is no C-K ordinal formula. This, although fundamental to the possibility of proving our completeness theorem does not form an actual step in the argument. Roughly speaking our argument will amount to this. The relation (9.3) implies that the system $P^{\underline{B}}$ is inconsistent and therefore that $P^{\underline{A}^*}$ is inconsistent, and indeed we can prove in P (and *a fortiori* in $P^{\underline{A}}$) that $\sim(x_0) \mathbin{\&} [x_0]$ implies the inconsistency of $P^{\underline{A}^*}$. On the other hand in $P^{\underline{A}}$ we can prove the consistency of $P^{\underline{A}^*}$. The inconsistency of $P^{\underline{B}}$ is proved by the Godel argument. Let us return to the details.

The axioms in $P^{\underline{B}}$ are those whose G.R's are of the form

$$\underline{B}(\lambda m n . m(\varpi(2,n), \varpi(3,n)), K, M_p, \underline{r})$$

Replacing $\underline{B}$ by $Suc(\underline{B})$ this becomes

$$Suc(\underline{B}, \lambda mn.\, m(\mathfrak{D}(2,n), \mathfrak{D}(3,n)), K, M_p, \underline{r})$$

$$\text{conv } K(\underline{B}(\lambda mn.\, m(\mathfrak{D}(2,n), \mathfrak{D}(3,n)), K, M_p, \underline{r})$$

$$\text{conv } \underline{B}(\lambda mn.\, m(\mathfrak{D}(2,n), \mathfrak{D}(3,n)), K, M_p, \underline{r})$$

$$\text{if } \underline{r} \text{ conv } 2\underline{p}+1$$

$$\text{conv } E(\underline{B}(\lambda mn.\, m(\mathfrak{D}(2,n), \mathfrak{D}(3,n)), K, M_p), \underline{p})$$

$$\text{if } \underline{r} \text{ conv } 2\underline{p}$$

When we remember the essential property of the formula E we see that the axioms of $P^{\underline{B}}$ include all formulae of the form

$$(\exists x_0)\, Proof_{P^{\underline{B}}}[x_0, f^{(q)}0] \supset \mathcal{F}$$

where q is the G.R. of the formula $\mathcal{F}$.

Let b be the G.R. of the formula $\mathfrak{A}$.

$$\sim(\exists y_0)(\exists x_0)\{Proof_{P^{\underline{B}}}[x_0, y_0] . Sb[z_0, z_0, y_0]\} \qquad (\mathfrak{A})$$

$Sb[x_0, y_0, z_0]$ is a particular recursion formula such that $Sb[f^{(l)}0, f^{(m)}0, f^{(n)}0]$ holds if and only if n is the G.R. of the result of substituting $f^{(m)}0$ for z_0 in the formula whose G.R. is l at all points where z_0 is free. Let p be the G.R. of the formula $\mathcal{L}$.

$$\sim(\exists y_0)(\exists x_0)\{Proof_{P^{\underline{B}}}[x_0, y_0] . Sb[f^{(b)}0, f^{(b)}0, y_0]\} \qquad (\mathcal{L})$$

Then we have as an axiom in $P^{\underline{B}}$

$$(\exists x_0)\ \mathrm{Prof}_{P^{\underline{B}}}[x_0, f^{(p)}0] \supset \mathcal{L}$$

and we can prove in P

$$(x_0)\ Sb\,[f^{(b)}0, f^{(b)}0, x_0] \supset x_0 = f^{(p)}0 \qquad (9.4)$$

since $\mathcal{L}$ is the result of substituting $f^{(b)}0$ for z_0 in $\mathfrak{A}$; whence

$$\sim(\exists y_0)\ \mathrm{Prof}_{P^{\underline{B}}}[y_0, f^{(p)}0] \qquad (9.5)$$

is provable in P. Using (9.4) again we see that $\mathcal{L}$ can be proved in $P^{\underline{B}}$. But if we can prove $\mathcal{L}$ in $P^{\underline{B}}$ then we can prove its provability in $P^{\underline{B}}$, the proof being in P ;i.e. we can prove

$$(\exists x_0)\ \mathrm{Prof}_{P^{\underline{B}}}[x_0, f^{(p)}0]$$

in P (since p is the G.R. of $\mathcal{L}$). But this contradicts (9.5), so that if $\sim\mathfrak{B}[f^{(a-1)}0]$ is true we can prove a contradiction in $P^{\underline{B}}$ or in $P^{\underline{A}^*}$. Now I assert that the whole argument up to this point can be carried through formally in the system P, in fact that if c be the G.R. of $\sim(0=0)$ then

$$\sim(a_0)\mathfrak{B}[a_0] \supset (\exists v_0)\ \mathrm{Prof}_{P^{\underline{A}^*}}[v_0, f^{(c)}0] \qquad (9.6)$$

is provable in P. I will not attempt to give any more detailed proof of this assertion.

The formula

$$(\exists x_0)\, \mathrm{Proof}_{P}\, \underline{A}^{*}[x_0, f^{(c)}0] \supset \sim(0=0) \qquad \mathbf{(9.7)}$$

is an axiom in $P^{\underline{A}}$. Combining (9.6),(9.7) we obtain $(x_0)\,\mathfrak{G}[x_0]$ in $P^{\underline{A}}$.

This completeness theorem as usual is of no value. Although it shows for instance that it is possible to prove Fermat's last theorem with Λ_P (if it is true) yet the truth of the theorem would really be assumed by taking a certain formula as an ordinal formula.

That Λ_P is not invariant may be proved easily by our general theorem; alternatively if follows from the fact that in proving our partial completeness theorem we never used ordinals higher than $\omega+1$. This fact can also be used to prove that Λ_P is not C-K invariant up to $\omega+2$.

10. The continuum hypothesis. A digression

The methods of § 9 may be applied to problems which are constructive analogues of the continuum hypothesis problem. The continuum hypothesis asserts that $2^{\aleph_0} = \aleph_1$, in other words that if ω_1 is the smallest ordinal α greater than ω such that a series with order type α cannot be put into one-one correspondence with the positive integers, then the ordinals less than ω_1 can be put into one-one correspondence with the subsets of the positive integers. To obtain a constructive analogue of this proposition we may replace the ordinals less than ω_1 either by the ordinal formulae, or by the ordinals represented by them; we may replace the subsets of the positive integers either by the computable sequences of figures 0, 1 or by the description numbers of the machines which compute these sequences. In the manner in which the correspondence is to be set up there is also more than one possibility. Thus even when we use only one kind of ordinal formula there is still great ambiguity as to what the constructive analogue of the continuum hypothesis should be. I shall prove a single result in this connection[23]. A number of others may be proved in the same way.

[23] A suggestion to consider this problem came to me indirectly from F. Bernstein. A related problem was suggested by P. Bernays.

We ask 'Is it possible to find a computable function of ordinal formulae determining a one-one correspondence between the ordinals represented by ordinal formulae and the computable sequences of figures 0, 1?'. More accurately 'Is there a formula $\underline{F}$ such that if $\underline{\Omega}$ is an ordinal formula and n a positive integer then $\underline{F}(\underline{\Omega}, \underline{n})$

is convertible to 1 or to 2, and such that $\underline{F}(\underline{\Omega}, \underline{n})$ conv $\underline{F}(\underline{\Omega}', \underline{n})$, for each positive integer n, if and only if $\underline{\Omega}$ and $\underline{\Omega}'$ represent the same ordinal?'. The answer is 'No', as will be seen to follow from this: there is no formula $\underline{F}$ such that $\underline{F}(\underline{\Omega})$ enumerates a certain sequence of integers (each being 1 or 2) when $\underline{\Omega}$ represents ω and enumerates another sequence when $\underline{\Omega}$ represents 0. If there is such an $\underline{F}$ then there is an a such that $\underline{F}(\underline{\Omega}, \underline{a})$ conv $\underline{F}(\mathfrak{Dt}, \underline{a})$ if $\underline{\Omega}$ represents ω but $\underline{F}(\underline{\Omega}, \underline{a})$ and $\underline{F}(\mathfrak{Dt}, \underline{a})$ are convertible to different integers (1 or 2) if $\underline{\Omega}$ represents 0. To obtain a contradiction from this we introduce a W.F.F. Gm not unlike Mg. If the machine $\mathcal{M}$ whose D.N. is n has printed 0 by the time the m th complete configuration is reached then $Gm(\underline{n}, \underline{m})$ conv $\lambda mn.\, m(n, I, 4)$ otherwise $Gm(\underline{n}, \underline{m})$ conv $\lambda pq.\, Al(4(P, 2p+2q), 3, 4)$. Now consider $\underline{F}(\mathfrak{Dt}, \underline{a})$ and $\underline{F}(\mathcal{L}im(Gm(\underline{n})), \underline{a})$. If $\mathcal{M}$ never prints 0 $\mathcal{L}im(Gm(\underline{n}))$ represents the ordinal ω. Otherwise it represents 0. Consequently these two formulae are convertible to one another if and only if $\mathcal{M}$ never prints 0. This gives us a means of telling of any machine whether it ever prints 0, which is impossible.

Results of this kind have of course no real relevance for the classical continuum hypothesis.

11. The purpose of ordinal logics.

Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two faculties[24], which we may call

[24] We are leaving out of account that most important faculty which distinguishes topics of interest from others; in fact we are regarding the function of the mathematician as simply to determine the truth of falsity of propositions.

which we may call intuition and ingenuity. The activity of the intuition consists in making spontaneous judgments which are not the result of conscious trains of reasoning. These judgments are often, but by no means invariably correct (leaving aside the question as to what is meant by 'correct'). Often it is possible to find some other way of verifying the correctness of an intuitive judgment. One may for instance judge that all positive integers are uniquely factorizable into primes; a detailed mathematical argument leads to the same result. It will also involve intuitive judgments, but they will be ones less open to criticism than the original judgment about factorization. I shall not attempt to explain this idea of 'intuition' any more explicitly.

The exercise of ingenuity in mathematics consists in aiding the intuition through suitable arrangements of propositions, and perhaps geometical figures or drawings. It is intended that when these are really well arranged validity of the intuitive steps which are required cannot seriously be doubted.

The parts played by these two faculties differ of course from occasion to occasion, and from mathematician to mathematician. This arbitrariness can be removed by the introduction of a formal logic. The necessity for using the intuition is then greatly reduced by setting down formal rules for carrying out inferences which are always intuitively valid. When working with a formal logic the idea of ingenuity takes a more definite shape. In general a formal logic will be framed so as to admit a considerable variety of possible steps in any stage in a proof. Ingenuity will then determine which steps are the more profitable for the purpose of proving a particular proposition. In pre-Gödel times it was thought by some that it would probably be possible to carry this program to such a point that all the intuitive judgments of mathematics could be replaced by a finite number of these rules. The necessity for intuition would then be entirely eliminated.

In our discussions, however, we have gone to the opposite extreme and eliminated not intuition but ingenuity, and this in spite of the fact that our aim has been in much the same direction. We have been trying to see how far it is possible to eliminate intuition, and leave only ingenuity. We do not mind how much ingenuity is required, and therefore assume it to be available in unlimited supply. In our metamathematical discussions we actually express this assumption rather differently. We are always able to obtain from the rules of a formal logic a method for enumerating the propositions proved by its means. We then imagine that all proofs take the form of a search through this enumeration for the theorem for which a proof is desired. In this way ingenuity is replaced by patience. In these heuristic discussions, however, it is better not to make this reduction.

Owing to the impossibility of finding a formal logic which will wholly eliminate the necessity of using intuition we naturally turn to 'non-constructive' systems of logic with which not all the steps in a proof are mechanical, some being intuitive. An example of a non-constructive logic is afforded by any ordinal logic. Wheh we have an ordinal logic we are in a position to prove number theoretic theorems by the intuitive steps of recognizing formulae as ordinal formulae, and the mechanical steps of carrying out conversions. What properties do we desire a non-constructive logic to have if we are to make use of it for the expression of mathematical proofs? We want it to be quite clear when a step makes use of intuition, and when it is purely formal. The strain put on the intuition should be a minimum. Most important of all, it must be beyond all reasonable doubt that the logic leads to correct results whenever the intuitive steps are correct[25]. It is also desirable that the logic be adequate

[25] This requirement is very vague. It is not of course intended that the criterion of the correctness of the intuitive steps be the correctness of the final result. The meaning becomes clearer if each intuitive step be regarded as a judgment that a particular proposition is true. In the case of an ordinal logic it is always a judgment that a formula is an ordinal formula, and this is equivalent to judging that a number theoretic proposition is true. In this case then the requirement is that the reputed ordinal logic be an ordinal logic.

for the expression of number theoretic theorems, in order that it may be used in metamathematical discussions (cf § 5).

Of the particular ordinal logics we have discussed Λ_P and Λ_H certainly will not satisfy us. In the case of Λ_H we are in no better position than with a constructive logic. In the case of Λ_P

(and for that matter also Λ_H) we are by no means certain that we shall never obtain any but true results, because we do not know whether all the number theoretic theorems provable in the system P are true. To take Λ_P as a fundamental non-constructive logic for metamathematical arguments would be most unsound. There remains the system of Church which is free of these objections. It is probably complete (although this would not necessarily mean much) and it is beyond reasonable doubt that it always leads to correct results[26].

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[26] This ordinal logic arises from a certain system C_0 in essentially the same way as Λ_P arose from P. By an argument similar to one occurring in § 8 we can show that the ordinal logic leads to correct results if and only if C_0 is valid; the validity of C_0 is proved in Church[1], making use of the results of Church and Rosser [1].

In the next section I propose to describe another ordinal logic, of a very different type, which is suggested by the work of Gentzen, and which should also be adequate for the formalization of number theoretic theorems. In particular it should be suitable for proofs of metamathematical theorems (cf § 5).

12. Gentzen type ordinal logics.

In proving the consistency of a certain system of formal logic Gentzen (Gentzen [1]) has made use of the principle of transfinite induction for ordinals less than ε_0, and suggested that it is to be expected that transfinite induction carried sufficiently far would suffice to solve all problems of consistency. Another suggestion to base systems of logic on transfinite induction has been made by Zermelo (Zermelo [1]). In this section I propose to show how this method of proof may be put into the form of a formal (non-constructive) logic, and afterwards to obtain from it an ordinal logic.

We could express the Gentzen method of proof formally in this way. Let us take the system P and adjoin to it an axiom $\mathfrak{A}_{\underline{\Omega}}$ with the intuitive meaning that the W.F.F. $\underline{\Omega}$ is an ordinal formula, whenever we feel certain that $\underline{\Omega}$ is an ordinal formula. This is a non-constructive system of logic which may easily be put into the form of an ordinal logic. By the method of § 6 we make correspond to the system of logic consisting of P with the axiom $\mathfrak{A}_{\underline{\Omega}}$ adjoined a logic formula $L_{\underline{\Omega}}$: $L_{\underline{\Omega}}$ is an effectively calculable function of $\underline{\Omega}$, and there is therefore a formula Λ_G^1 such that $\Lambda_G^1(\underline{\Omega})$ conv $\underline{\Omega}$ for each formula $\underline{\Omega}$. Λ_G^1 is certainly not an ordinal logic unless P is valid, and therefore consistent. This formalization of Gentzen's idea would therefore not be applicable for the problem with which Gentzen himself was concerned, for he was proving the consistency of a system weaker then P. However, there are other ways in which the Gentzen method of proof can be formalized. I shall explain one, beginning by describing a certain system of symbolic logic.

The symbols of the calculus are f, x, $'$, $_{|}$, 0, S, R, Γ, Δ, E, $|$, $\odot$, $!$, $($, $)$, $\overset{|}{=}$, and the comma ','. We use capital German letters to stand for variable or undetermined sequences of these symbols.

It is to be understood that the relations that we are about to define hold only when compelled to do so by the conditions we lay down. The conditions should be taken together as a simultaneous inductive definition of all the relations involved.

Suffixes

$_{|}$ is a suffix. If $\mathfrak{S}$ is a suffix then $\mathfrak{S}_{|}$ is a suffix.

Indices

$'$ is an index. If $\mathfrak{I}$ is an index then $\mathfrak{I}'$ is an index.

Numerical variables

If $\mathfrak{S}$ is a suffix then $x\mathfrak{S}$ is a numerical variable.

Functional variables

If $\mathfrak{S}$ is a suffix and $\mathfrak{I}$ is an index then $f\mathfrak{S}\mathfrak{I}$ is a functional variable of index $\mathfrak{I}$.

Arguments

$(,)$ is an argument of index $'$. If $(\mathfrak{A})$ is an argument of index $\mathfrak{I}$ and $\mathfrak{T}$ is a term then $(\mathfrak{A}\mathfrak{T},)$ is an argument of index $\mathfrak{I}'$.

Numerals

0 is a numeral.

If $\mathfrak{n}$ is a numeral then $S(,\mathfrak{n},)$ is a numeral.

In metamathematical statements we shall denote the numeral in which S occurs r times by $S^{(r)}(,0,)$.

Expressions of given index

A functional variable of index $\mathfrak{J}$ is an expression of index $\mathfrak{J}$.

R, S are expressions of index $'''$, $''$ respectively.

If $\mathfrak{n}$ is a numeral then it is also an expression of index $'$.

Suppose $\mathfrak{Y}$ is an expression of index $\mathfrak{J}$, $\mathfrak{Z}$ one of index $\mathfrak{J}'$ and $\mathfrak{X}$ one of index $\mathfrak{J}'''$; then $(\Pi\,\mathfrak{Y})$ and $(\Delta\,\mathfrak{Y})$ are expressions of index $\mathfrak{J}$, whilst $(E\,\mathfrak{Y})$ and $(\mathfrak{Y}\odot\mathfrak{X})$ and $(\mathfrak{Y}/\mathfrak{Z})$ and $(\mathfrak{Y}\,!\,\mathfrak{Z}\,!\,\mathfrak{X})$ are expressions of index $\mathfrak{J}'$.

Function constants

An expression of index $\mathfrak{J}$ in which no functional variable occurs is a function constant of index $\mathfrak{J}$. If in addition R do not occur the expression is called a primitive function constant.

Terms

0 is a term.

Every numerical variable is a term.

If $\mathfrak{Y}$ is an expression of index $\mathfrak{J}$ and $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$ then $\mathfrak{Y}(\mathfrak{A})$ is a term.

Equations

If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are terms then $\mathfrak{T}_1 = \mathfrak{T}_2$ is an equation.

Provable equations

We define what is meant by the provable equations relative to a given set of equations as axioms.

(a) The provable equations include all the axioms. The axioms are of the form of equations in which the symbols Γ , Δ , E , $|$, $\odot$, $!$ do not appear.

(b) If $\mathfrak{Y}$ is an expression of index $\mathfrak{J}''$ and $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$ then

$$(\Gamma\mathfrak{Y})(\mathfrak{A}\, x_{1}, x_{11},) = \mathfrak{Y}(\mathfrak{A}\, x_{11}, x_{1},)$$

is a provable equation.

(c) If $\mathfrak{Y}$ is an expression of index $\mathfrak{J}'$, and $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$, then

$$(\Delta\mathfrak{Y})(\mathfrak{A}\, x_{1},) = \mathfrak{Y}(, x_{1}\mathfrak{A})$$

is a provable equation.

(d) If $\mathfrak{Y}$ is an expression of index $\mathfrak{J}$, and $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$, then

$$(E\mathfrak{Y})(\mathfrak{A}\, x_{1},) = \mathfrak{Y}(\mathfrak{A})$$

is a provable equation.

(e) If $\mathfrak{G}$ is an expression of index $\mathfrak{J}$ and $\mathfrak{H}$ is one of index $\mathfrak{J}'$, and $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$, then

$$(\mathfrak{G} \mid \mathfrak{H})(\mathfrak{A}) = \mathfrak{H}(\mathfrak{A}\,\mathfrak{G}(\mathfrak{A}),)$$

is a provable equation.

(f) If $\mathfrak{M}$ is an expression of index $'$ then $\mathfrak{M}(,) = \mathfrak{M}$ is a provable equation.

(g) If $\mathfrak{G}$ is an expression of index $\mathfrak{J}$ and $\mathfrak{R}$ one of index $\mathfrak{J}'''$, and $(\mathfrak{A})$ an argument of index $\mathfrak{J}'$, then

$$(\mathfrak{G} \odot \mathfrak{R})(\mathfrak{A}\,0,) = \mathfrak{G}(\mathfrak{A})$$

and

$$(\mathfrak{G} \odot \mathfrak{R})(\mathfrak{A}\,S(,x_1,),) = \mathfrak{R}(\mathfrak{A}\,x_1,S(,x_1,),(\mathfrak{G} \odot \mathfrak{R})(\mathfrak{A}\,x_1,),)$$

are provable equations. If in addition $\mathfrak{H}$ is an expression of index $\mathfrak{J}'$ and

$$R(,\mathfrak{H}(\mathfrak{A}\,S(,x_1,),),x_1,) = 0$$

is provable then

$$(\mathfrak{G}!\mathfrak{R}!\mathfrak{H})(\mathfrak{A}\,S(,x_1,),) = \mathfrak{R}(\mathfrak{A}\,\mathfrak{H}(\mathfrak{A}\,S(,x_1,),),S(,x_1,),$$
$$(\mathfrak{G}!\mathfrak{R}!\mathfrak{H})(\mathfrak{A}\,\mathfrak{H}(\mathfrak{A}\,S(,x_1,),),),)$$

and

$$(\mathfrak{G} ! \mathfrak{K} ! \mathfrak{L})(\mathfrak{A}\, 0,) = \mathfrak{G}(\mathfrak{A})$$

are provable.

(h) If $\mathfrak{T}_1 = \mathfrak{T}_2$ and $\mathfrak{T}_3 = \mathfrak{T}_4$ are provable where $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3$, and $\mathfrak{T}_4$ are terms then $\mathfrak{T}_4 = \mathfrak{T}_3$ and the result of substituting $\mathfrak{T}_3$ for $\mathfrak{T}_4$ at any particular occurrence in $\mathfrak{T}_1 = \mathfrak{T}_2$ are provable equations.

(i) If $\mathfrak{T}_1 = \mathfrak{T}_2$ is a provable equation then the result of substituting any term for a particular numerical variable throughout this equation is provable.

(j) Suppose that $\mathfrak{G}$, $\mathfrak{G}_1$ are expressions of index $\mathfrak{J}'$, that $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$ not containing the numerical variable $\mathfrak{x}$ and that $\mathfrak{G}(\mathfrak{A}\, 0,) = \mathfrak{G}_1(\mathfrak{A}\, 0,)$ is provable. Also suppose that if we add $\mathfrak{G}(\mathfrak{A}\, \mathfrak{x},) = \mathfrak{G}_1(\mathfrak{A}\, \mathfrak{x},)$ to the axioms and restrict (i) so that it can never be applied to the numerical variable $\mathfrak{x}$ then

$$\mathfrak{G}(\mathfrak{A}\, S(,\mathfrak{x},),) = \mathfrak{G}_1(\mathfrak{A}\, S(,\mathfrak{x},),)$$

becomes a provable equation; in the hypothetical proof of this equation this rule (j) itself may be used provided that a different variable is chosen to take the part of $\mathfrak{x}$.

Under these conditions $\mathfrak{G}(\mathfrak{A}\, \mathfrak{x},) = \mathfrak{G}_1(\mathfrak{A}\, \mathfrak{x},)$ is a provable equation.

(k) Suppose that $\mathfrak{G}$, $\mathfrak{G}_1$, $\mathfrak{L}$ are expressions of index $\mathfrak{J}'$, that $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$ not containing the numerical variable $\mathfrak{x}$ and that $\mathfrak{F}(\mathfrak{A}\,0,) = \mathfrak{F}_1(\mathfrak{A}\,0,)$ and $R(,\mathfrak{G}(\mathfrak{A}\,S(,\mathfrak{x},),),S(,\mathfrak{x},),) = 0$ are provable equations. Suppose also that if we add

$$\mathfrak{F}(\mathfrak{A}\,\mathfrak{G}(\mathfrak{A}\,S(,\mathfrak{x},),),) = \mathfrak{F}_1(\mathfrak{A}\,\mathfrak{G}(\mathfrak{A}\,S(,\mathfrak{x},),),)$$

to the axioms, and again restrict (i) so as not to apply to $\mathfrak{x}$ then

$$\mathfrak{F}(\mathfrak{A}\,\mathfrak{x},) = \mathfrak{F}_1(\mathfrak{A}\,\mathfrak{x},) \qquad (12.1)$$

becomes a provable equation; in the hypothetical proof of (12.1) the rule (k) may be used if a different variable takes the part of $\mathfrak{x}$.

Under these conditions (12.1) is a provable equation.

We have now completed the definition of a provable equation relative to a given set of axioms. Next we shall show how to obtain an ordinal logic from this calculus. The first step is to set up a correspondence between some of the equations and number theoretic theorems, in other words to show how they can be interpreted as number theoretic theorems. Let $\mathfrak{F}$ be a primitive function constant of index 111. $\mathfrak{F}$ describes a certain primitive recursive function $\phi(m,n)$, determined by the condition that for all m, n the equation

$$\mathfrak{F}(,S^{(m)}(,0,),S^{(n)}(,0,),) = S^{(\phi(m,n))}(,0,)$$

shall be provable without using the axioms (a). Suppose also that $\mathfrak{G}$ is an expression of index 1. Then to the equation

$$\mathfrak{F}(,x_1,\mathfrak{G}(,x_1,),) = 0$$

we make correspond the number theoretic theorem which asserts that for each natural number m there is a natural number n such that $\varphi(m,n) = 0$. (The circumstances that there is more than one equation to represent each number theoretic theorem could be avoided by a trivial modification of the calculus.)

Now let us suppose some definite method is chosen for describing the sets of axioms by means of positive integers, the null set of axioms being described by the integer 1. By an argument used in § 6 there is a W.F.F. Σ such that if r is the integer describing a set A of axioms then $\Sigma(\underline{r})$ is a logic formula enabling us to prove just those number theoretic theorems which are associated with equations provable with the above described calculus, the axioms being just those described by the number r.

I shall show two ways in which the construction of the ordinal logic may be completed.

In the first method we make use of the theory of general recursive functions (Kleene [2]). Let us consider all equations of the form

$$R(,S^{(m)}(,0,)),S^{(n)}(,0,),) = S^{(p)}(,0,) \qquad (12.2)$$

which are obtainable from the axioms by the use of rules (h), (i). It is a consequence of the theorem of equivalence of λ-definable and general recursive function (Kleene [3]) that if $r(m,n)$ is any λ-definable function of two variables then we can choose the axioms so that (12.2) with $p = r(m,n)$ is obtainable in this way for each pair of natural numbers m, n, and no equation of the form

$$S^{(m)}(,0,) = S^{(n)}(,0,) \qquad (m \neq n) \tag{12.3}$$

is obtainable. In particular this is the case if $r(m,n)$ is defined by the condition that

$$\underline{\Omega}(\underline{m},\underline{n}) \text{ conv } S(\underline{p}) \text{ implies } p = r(m,n)$$
$$r(0,n) = 0 \text{ all } n > 0, \quad r(0,0) = 2$$

where $\underline{\Omega}$ is an ordinal formula. There is a method for obtaining the axioms given the ordinal formula, and consequently a formula Rec such that for any ordinal formula $\underline{\Omega}$, $Rec(\underline{\Omega})$ conv $\underline{m}$ where m is the integer describing the set of axioms corresponding to $\underline{\Omega}$. Then the formula

$$\Lambda_G^2 \rightarrow \lambda \omega . \, \Sigma(Rec(\omega))$$

is an ordinal logic. Let us leave the proof of this aside for the present.

Our second ordinal logic is to be constructed by a method not unlike the one we used in constructing Λ_P. We begin by assigning ordinal formulae to all sets of axioms satisfying certain conditions. For this purpose we again consider that part of the calculus which is obtained by restricting 'expressions' to be functional variables or R or S and restricting the meaning of 'term' accordingly; the new provable equations are given by conditions (a), (h), (i), together with an extra condition (l)

(1) The equation

$$R(,0,S(,x,,),,)=0$$

is provable.

We could design a machine which would obtain all equations of the form (12.2), with $m \neq n$, provable in this sense, and all of the form (12.3), except that it would cease to obtain any more equations when it had once obtained one of the latter 'contradictory' equations. From the description of the machine we obtain a formula $\underline{\Omega}$ such that

$\underline{\Omega}(\underline{m},\underline{n})$ conv 2 if $\quad R(,S^{(m-1)}(,0,),\ S^{(n-1)}(,0,),,)=0$

is obtained by the machine

$\underline{\Omega}(\underline{m},\underline{n})$ conv 1 if $\quad R(,S^{(n-1)}(,0,),\ S^{(m-1)}(,0,),,)=0$

is obtained by the machine

$\underline{\Omega}(\underline{m},\underline{m})$ conv 3 always.

The formula $\underline{\Omega}$ is an effectively calculable function of the set of axioms, and therefore also of m : consequently there is a formula M such that $M(\underline{m})$ conv $\underline{\Omega}$ when m describes the set of axioms. Now let Cm be a formula such that if b is the G.R. of a formula $M(\underline{m})$ then $Cm(\underline{b})$ conv $\underline{m}$, but otherwise $Cm(\underline{b})$ conv 1. Let

$$\underline{\Lambda}_G^3 \rightarrow \lambda \omega a.\, \Gamma(\lambda u.\, \Sigma(Cm(Tn(\omega,u)),a)$$

Then $\Lambda_G^3(\underline{\Omega},\underline{A})$ conv 2 if and only if $\underline{\Omega}$ conv $M(\underline{m})$ where m describes a set of axioms which, taken with our calculus, suffices to prove the equation which is, roughly speaking, equivalent to '$\underline{A}$ is dual'. To prove that Λ_G^3 is an ordinal logic it suffices to prove that the calculus with the axioms described by n proves only true number theoretic theorems when $\underline{\Omega}$ is an ordinal formula. This condition on m may also be expressed in this way. Let us put $m << n$ if we can prove $R(,S^{(m)}(,0,),S^{(n)}(,0,),)=0$ with (a), (h), (i), (l): the condition is that $m << n$ be a well ordering of the natural numbers and that no contradictory equation (12.3) be provable with the same rules (a), (h), (i), (l). Let us say that such a set of axioms is <u>admissible</u>. Λ_G^3 is an ordinal logic if the calculus leads to none but true number theoretic theorems when an admissible set of axioms is used.

In the case of Λ_G^2, $Rec(\underline{\Omega})$ describes an admissible set of axioms whenever $\underline{\Omega}$ is an ordinal formula. Λ_G^2 will therefore be an ordinal logic if the calculus leads to correct results when admissible axioms are used.

To prove that admissible axioms have this property I shall not attempt to do more than show how interpretations can be given to the equations of the calculus so that the rules of inference (a) - (k) become intuitively valid methods of deduction, and so that the interpretation agrees with our convention regarding number theoretic theorems.

Each expression is the name of a function, which may be only partially defined. The expression S corresponds simply to the successor function. If $\mathfrak{G}$ is either R or a functional variable and is of index $\mathfrak{J}$ ($p+1$ symbols in the index) then it corresponds to a function g of p natural numbers defined as follows. If

$$\mathfrak{G}(,S^{(r_1)}(,0,),S^{(r_2)}(,0,),\ldots,S^{(r_p)}(,0,),) = S^{(l)}(,0,)$$

is provable by the use of (a), (h), (i), (l) only, then $g(r_1,\ldots,r_p)$ has the value l. It may not be defined for all arguments, but its value is always unique, for otherwise we could prove a 'contradictory' equation and $M(\underline{m})$ would then not be an ordinal formula. The functions corresponding to the other expressions are essentially defined by (b) - (f). For example if g is the function corresponding to $\mathfrak{G}$ and g' that corresponding to $(\Gamma\,\mathfrak{G})$ then

$$g'(r_1,r_2,\ldots,r_p,l,m) = g(r_1,r_2,\ldots,r_p,m,l)$$

The values of the functions are clearly unique (when defined at all) if given by one of (b) - (e). The case (f) is less obvious since the function defined appears also in the definiens. I shall not treat the case of $(\mathfrak{G}\ominus\mathfrak{H})$ as this is the well known definition by primitive recursion, but let us show the values of the function corresponding to $(\mathfrak{G}!\mathfrak{R}!\mathfrak{H})$ are unique. Without loss of generality we may suppose that $(\mathfrak{R})$ is of index $/$. We have then to show that if $h(m)$ is the function corresponding to $\mathfrak{H}$ and $r(m,n)$ that corresponding to $\mathfrak{R}$, and $k(u,v,w)$ a given function and a a given natural number then the equations

$$l(0) = a \qquad \alpha)$$

$$l(m+1) = k(h(m+1), m+1, l(h(m+1))) \qquad \beta)$$

do not ever assign two different values for the function $l(m)$. Consider those values of r for which we obtain more than one value of $l(r)$, and suppose that there is at least one such. Clearly 0 is not one for $l(0)$ can only be defined by $\alpha)$. As the relation $<<$ is a well ordering there is an integer r_0 such that $r_0 > 0$, $l(r_0)$ is not unique, and if $s \neq r_0$ and $l(s)$ is not unique then $r_0 << s$. Putting $s = h(r_0)$ we find also $s << r_0$ which is impossible. There is therefore no value for which we obtain more than one value for the function $l(r)$.

Our interpretation of expressions as functions give us an immediate interpretation for equations with no numerical variables. In general we interpret an equation with numerical variables as the conjunction of all equations obtainable by replacing the variables by numerals. With this interpretation (h), (i) are seen to be valid methods of proof. In (j) the provability of

$$\mathfrak{G}(\mathfrak{A}\,S(,x_1,),) = \mathfrak{G}_1(\mathfrak{A}\,S(,x_1,),)$$

when $\mathfrak{G}(\mathfrak{A}\,x_1,) = \mathfrak{G}_1(\mathfrak{A}\,x_1,)$ is assumed to be interpreted as meaning that the implication between these equations holds for all substitutions of numerals for x_1. To justify this one should satisfy oneself that these implications always hold when the hypothetical proof can be carried out. The rule of procedure (j) is now seen to be simply mathematical induction. The rule (k) is a form of transfinite induction. In proving the validity of (k) we may again suppose $(\mathfrak{A})$ is of index 1. Let $r(m,n)$, $g(m)$, $g_1(m)$, $h(n)$ be the functions corresponding respectively to R, $\mathfrak{G}$, $\mathfrak{G}_1$, $\mathfrak{H}$.

We shall prove that if $g(0)=g_1(0)$ and $r(h(n),n)=0$ for each positive integer n and $g(n+1)=g_1(n+1)$ whenever $g(h(n+1))=g_1(h(n+1))$ then $g(n)=g_1(n)$ for each natural number n. We consider the class of integers n for which $g(n)=g_1(n)$ is not true. If the class is not void it has a positive member n_0 which precedes all other members in the well ordering $<<$. But $h(n_0)$ is another member of the class, for otherwise we should have $g(h(n_0))=g_1(h(n_0))$ and therefore $g(n_0)=g_1(n_0)$ i.e. n_0 would not be in the class. This implies $n_0 << h(n_0)$ contrary to $r(h(n_0),n_0)=0$. The class is therefore void.

It should be noticed that we do not really need to make use of the fact that $\underline{\Omega}$ is an ordinal formula. It suffices that $\underline{\Omega}$ should satisfy conditions (a) - (e) (p.29) for ordinal formulae, and in place of (f) satisfy (f').

(f') There is no formula $\underline{T}$ such that $\underline{T}(\underline{n})$ is convertible to a formula representing a positive integer for each positive integer n, and such that $\underline{\Omega}(\underline{T}(\underline{n}),\underline{n})$ conv 2, for each positive integer n for which $\underline{\Omega}(\underline{n},\underline{n})$ conv 3.

The problem as to whether a formula satisfies conditions (a) - (e), (f') is number theoretic. If we use formulae satisfying these conditions instead of ordinal formulae with Λ_G^3 we have a non-constructive logic with certain advantages over ordinal logics. The intuitive judgments that must be made are all judgments of the truth of number theoretic theorems. We have seen in § 9 that the connection of ordinal logics with the classical theory of ordinals is quite superficial. There seem to be good reasons therefore for giving attention to ordinal formulae in this modified sense.

The ordinal logic Λ_G^3 appears to be adequate for most purposes. It should for instance be possible to carry out Gentzen's proof of consistency of number theory, or the proof of the uniqueness of the normal form of a well-formed formula (Church and Rosser [1]) with our calculus and a fairly simple set of axioms. How far this is the case can of course only be determined by experiment.

One would prefer that a non-constructive system of logic based on transfinite induction were rather simpler than the one we have described. In particular it would seem that it should be possible to eliminate the necessity of stating explicitly the validity of definitions by primitive recursions, as this principle itself can been shown to be valid by transfinite induction. It is possible to make such modifications in the system, even in such a way that the resulting system is still complete, but no real advantage is gained by doing so. The effect is always, so far as I know, to restrict the class of formulae provable with a given set of axioms, so that we obtain no theorems but trivial restatements of the axioms. We have therefore to compromise between simplicity and comprehensiveness.

Index of definitions

No attempt is being made to list underlined formulae as their meanings are not always constant throughout the paper. Abbreviations for definite well-formed formulae are listed alphabetically.

Miscellaneous (in order of appearance)

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